The arctic product-mix market: unifying revenue and welfare

Simon Finster* Paul W. Goldberg† Edwin Lock‡

Abstract

In arctic product-mix markets, buyers can express preferences across multiple divisible goods in conjunction with a monetary budget. Like in Fisher markets, bidders are guaranteed to receive goods that maximise their bang-per-buck. But unlike Fisher markets, not every buyer necessarily spends their entire budget. Instead, buyers reject a potential purchase if prices are too high. This leads to a novel coincidence of competitive equilibrium and optimal revenue in our market. We show that competitive equilibrium prices are unique. Moreover, these prices maximise not only social welfare but also revenue, and we present an algorithm to identify these prices.

1 Introduction

Finding equilibrium prices in multi-good markets has been of great interest to economists for centuries. The divisible goods market by Léon Walras [15] and its important special case, the Fisher market [4], have received a great deal of attention, especially following Arrow and Debreu's result on the existence of a general market equilibrium [2]. Economists and computer scientists have since developed much algorithmic machinery to compute market equilibria (e.g. [1, 5, 6, 7, 9, 10, 14]), and efficient algorithms have been described for both Walras and Fisher markets for the linear utilities case and beyond. However, in many applications, these markets have one important short-coming: demand is scale-invariant, that is buyers may demand goods at arbitrarily high prices.

In this paper, we consider a market of multiple divisible goods with richer preferences that are scale-*variant*. This property of demand makes it possible to consider revenue, or profit, as an objective for the seller: if not all supply is sold by default, it may be profitable to raise prices and withhold some supply from the market instead. In Fisher markets, this is not a viable objective because bidders spend their entire budget at any prices, resulting in identical revenue for the seller. Therefore, our contribution starts with fundamentals and reveals a surprising and desirable feature of this market, before developing algorithmic results. We prove the existence of unique market equilibrium prices, that *simultaneously* maximise the seller's revenue. Because the welfare theorems hold in our setting, these prices have the desirable property of being efficient *and* optimal.

The market we consider is an important special case of the arctic product-mix auction market first proposed by Paul Klemperer [13] for the government of Iceland.[1] In this market, buyers express their demand preferences by submitting bids that consist of a monetary budget and a bid price for each good to the seller. At a market price *above the stated bid price*, a buyer *rejects the good*. At market prices below the stated bid prices, a buyer demands the goods that yield the highest 'bang-per-buck', i.e. the highest ratio of value to price, a feature in common with the Fisher market. Through the ability to reject a good, however, our market generalises the standard Fisher market.

*Nuffield College and Department of Economics, University of Oxford, simon.finster@nuffield.ox.ac.uk.
†Department of Computer Science, University of Oxford, paul.goldberg@cs.ox.ac.uk
‡Nuffield College and Department of Economics, University of Oxford, edwin.lock@economics.ox.ac.uk

[1] The arctic product-mix auction is a variant of the original product-mix auction developed for the Bank of England by Paul Klemperer [11, 12]. See also [3]. Klemperer also chose the name 'arctic' to contrast with the fact that the original (but not the arctic) product-mix auction has close connections to tropical geometry. In the general arctic product-mix auction, the seller can additionally choose cost functions to fine-tune its preferences, while in our model, we assume the seller's costs to be zero. See [8] for some discussion of the general case.

The market. We consider the following market. A number of buyers compete for the supply of multiple units of different goods with zero reserve prices. Each buyer ('he') has a given amount of money available to spend on any of the goods, his budget, and per-unit values for each good. A combination of budget and per-unit values for each good is called a bid. Because the seller processes each bid independently, it is without loss of generality to assume that each buyer submits a single bid, i.e. has a single budget-value combination. The seller ('she') sets a uniform market price for each good, i.e. each buyer who demands a given good pays this price per allocated unit. Given a market price vector (subsequently simply called 'price vector'), each buyer demands only those goods that give him the maximal 'bang-per-buck', that is the maximal ratio of value to price among all goods. If a buyer demands a single good, he demands his entire budget's worth of this good. If a buyer demands multiple goods, he demands quantities that, multiplied by prices, make up his entire budget. If a buyer's values are strictly lower than prices, a buyer demands nothing. If a buyer is indifferent between one or multiple goods and receiving nothing, he demands quantities that, multiplied by prices, make up weakly less than his budget.

Results. In setting the market prices, the seller can pursue a variety of objectives. The focus of our contribution is on the objectives of maximising revenue, achieving competitive equilibrium, and optimising efficiency and social welfare. Our model assumes complete information of all agents in the markets and truthful submission of valuations. Thus, social welfare is defined with respect to the submitted valuations. If the seller aims to maximise efficiency, it is intuitive that she wants to sell her entire supply; we prove this in Section 6. However, if the seller aims to maximise her revenue, she may want raise prices and keep some of her supply to achieve higher revenue. As our main result, we prove that our market admits a unique price vector $\boldsymbol{p}^*$ that unifies our two objectives: $\boldsymbol{p}^*$ clears the market and supports a competitive equilibrium (Section 6), and it maximises revenue (Section 4). In Section 7, we also present an algorithm to find these prices. While we show that market-clearing prices are unique in our market, the same does not hold for revenue-optimal prices. Our prices $\boldsymbol{p}^*$ are, nevertheless, bidder-optimal revenue-maximising prices in the sense that allocated quantities are maximised.

We start by establishing a common representation of the buyers' utility, implying that our market is 'well-behaved', that is the first and second welfare theorem hold. We introduce the 'feasible region' in price space, which consists of the set of prices at which demand does not exceed supply. At prices not contained in the region, market demand exceeds the supply of at least one good. We demonstrate that the feasible region has the lower semi-lattice property, guaranteeing the existence of element-wise minimal prices. We proceed to show that at these prices, the seller's revenue is maximised. Intuitively, the seller's revenue only depends on collecting the budgets of those bids that accept to buy at the market prices she sets. Thus, setting prices as low as possible while not violating supply must be revenue-optimal. However, it is not immediately obvious that markets clear at the element-wise minimal feasible price vector. We adopt a tree-based price reduction procedure that bears conceptual similarities to the approach in [9] and allows us to reduce any given price vector that is not market-clearing. This immediately suggests an algorithm to compute market-clearing prices. It is an open question whether our algorithm admits a polynomial-time running time analysis; we expect our current exponential bound to be loose. Finally, we demonstrate by means of our price reduction procedure that there exist unique market-clearing prices, implying that our algorithm finds competitive equilibrium and revenue-optimal prices in one.

Applications. The market we consider is a special case of an auction originally developed for the government of Iceland (see [13]). The government planned to use this auction market to exchange blocked accounts for other financial assets, e.g. cash or bonds with different properties, which were available only in limited quantities. The auction was especially fitting for this use because buyers could express a budget, as well as trade-offs between different assets through bid prices. Other related applications may include debt restructuring and the (re-)division of firms between shareholders [13].

Our market may be applicable to ad auctions. When advertising companies compete for web space (goods) to display their ads, the decision which publisher to choose and bid for is difficult. It may be intuitive, however, to choose an advertising budget and state demand in terms of 'limit market prices' for multiple, distinct goods, below which the seller, or the market platform, allocates those good with the highest bang-per-buck. This is not only conceptually easier for the advertiser (buyer), but also practical and feasible through our results and algorithm for the arctic product-mix setting.

Preliminaries. Let $[n] := \{1, \ldots, n\}$ denote n distinct goods. We often work with a notional *reject good*, denoted by 0, and let $[n]_0 := \{0, \ldots, n\}$. A *bundle* of goods, typically denoted by $\boldsymbol{x}$ or $\boldsymbol{y}$ in this paper, is a vector in $\mathbb{R}^n$ whose i-th entry denotes the *quantity* of good i. A price vector $\boldsymbol{p} \in \mathbb{R}^n$ has a price entry for each of the n goods. We write $\boldsymbol{p} \leq \boldsymbol{q}$ when the inequality holds element-wise. The price p_0 of the reject good is defined as 1. The dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as $\boldsymbol{a} \cdot \boldsymbol{b}$. For any function $f : A \times B \to \mathbb{R}^n$, we use the implicit summation $f(A', b) = \sum_{a \in A'} f(a, b)$, $f(a, B') = \sum_{b \in B'} f(a, b)$, and $f(A', B') = \sum_{a \in A', b \in B'} f(a, b)$, for any $A' \subseteq A$ and $B' \subseteq B$.

2 Market model

Each bidder submits a finite list $\mathcal{B}$ of arctic bids. The seller treats each bid independently, so we may assume without loss of generality that each bidder submits a single bid. We assume agents have full knowledge of the market and behave as price takers. A seller has a given supply $\boldsymbol{s} \in \mathbb{R}^n_+$ of multiple, divisible goods and wishes to allocate some subset of this supply among a finite set of buyers. The notional reject good is available with infinite supply $s_0 = \infty$. The seller sets a market price vector $\boldsymbol{p} \in \mathbb{R}^n_+$ that induces an allocation according to bidders' demand.

Demand. A bid consists of an n-dimensional vector $\boldsymbol{b} \in \mathbb{R}^n_+$ and a budget $\beta(\boldsymbol{b}) \in \mathbb{R}_+$. For each bid $\boldsymbol{b}$, we can understand b_i as the amount that $\boldsymbol{b}$ is willing to spend per unit of good i, that is $\boldsymbol{b}$'s per-unit value of good i. When working with the notional reject good 0, we identify a bid vector $\boldsymbol{b}$ with the $n+1$-dimensional vector obtained by adding a 0-th entry of value 1. Without loss of generality, we assume that there is at least one bid $\boldsymbol{b}$ with $b_i > 0$ for every good $i \in [n]$.[2]

Suppose the seller sets market prices $\boldsymbol{p}$. A bid $\boldsymbol{b}$ is *rejected* at $\boldsymbol{p}$ if $b_i < p_i$ for all goods $i \in [n]$. Otherwise, the bid *demands a good* $i \in [n]$ that maximises $\frac{b_i}{p_i}$ at price $\boldsymbol{p}$. Recalling that $b_0 = 1 = p_0$, we say that $\boldsymbol{b}$ demands good $i \in [n]_0$ if $i \in \arg\max_{i \in [n]_0} \frac{b_i}{p_i}$, and receiving the 'reject' good is equivalent to the bid being rejected. The demand of a single 'arctic' bid is illustrated in Fig. 1. We let $I_{\boldsymbol{b}}(\boldsymbol{p})$ denote the goods bid $\boldsymbol{b}$ demands at $\boldsymbol{p}$, i.e. $I_{\boldsymbol{b}}(\boldsymbol{p}) = \arg\max_{i \in [n]_0} \frac{b_i}{p_i}$. If $I_{\boldsymbol{b}}(\boldsymbol{p})$ contains more than one good, we say that $\boldsymbol{b}$ is *indifferent* between these goods at $\boldsymbol{p}$. A bid may be indifferent between demanding goods and being rejected when $\max_{i \in [n]_0} \frac{b_i}{p_i} = 1$. At given market prices $\boldsymbol{p}$, each bid spends its budget $\beta(\boldsymbol{b})$ on a combination of goods $I_{\boldsymbol{b}}(\boldsymbol{p})$. Hence, we define its *demand correspondence* $D_{\boldsymbol{b}}$ as the set of bundles $D_{\boldsymbol{b}}(\boldsymbol{p}) := \left\{\boldsymbol{x} \in \mathbb{R}^n_+ \mid \boldsymbol{x} \cdot \boldsymbol{p} = \beta(\boldsymbol{b}) \text{ and } x_i > 0 \implies i \in I_{\boldsymbol{b}}(\boldsymbol{p})\right\}$. The demand correspondence $D_{\mathcal{B}}$ of a bid list $\mathcal{B}$ is then given by the Minkowski sum $D_{\mathcal{B}}(\boldsymbol{p}) = \left\{\sum_{\boldsymbol{b} \in \mathcal{B}} \boldsymbol{x}^{\boldsymbol{b}} \mid \boldsymbol{x}^{\boldsymbol{b}} \in D_{\boldsymbol{b}}(\boldsymbol{p})\right\}$.

Expenditure and allocation. An expenditure $e : \mathcal{B} \times [n]_0 \to \mathbb{R}_+$ expresses, for all bids $\boldsymbol{b} \in \mathcal{B}$ and all goods $i \in [n]_0$, the amount $e(\boldsymbol{b}, j)$ that bid $\boldsymbol{b}$ spends on good i. For each expenditure, we can define the corresponding *allocation* $\pi : \mathcal{B} \times [n]_0 \to \mathbb{R}_+$ denoting the amount $\pi(\boldsymbol{b}, j) = \frac{e(\boldsymbol{b}, j)}{p_j}$ of good j that bid $\boldsymbol{b}$ receives. For any subset $\mathcal{B}' \subseteq \mathcal{B}$ and $I \subseteq [n]_0$, we recall the shorthand $e(\mathcal{B}', i) := \sum_{\boldsymbol{b} \in \mathcal{B}'} e(\boldsymbol{b}, i)$, $e(\boldsymbol{b}, I) := \sum_{i \in I} e(\boldsymbol{b}, i)$, and $e(\mathcal{B}', I) := \sum_{\boldsymbol{b} \in \mathcal{B}', i \in I} e(\boldsymbol{b}, i)$ (and similarly for allocations π). Given an expenditure e, we say that good $i \in [n]$ is *sated* if $e(\mathcal{B}, i) = p_i s_i$, and *unsated* if $e(\mathcal{B}, i) < p_i s_i$.

Definition 1 (Feasibility and validity)**.** Let $e : \mathcal{B} \times [n]_0 \to \mathbb{R}_+$ be an expenditure. Then e is:

(i) *Supply-feasible* at prices $\boldsymbol{p}$ if supply weakly exceeds the allocation for all goods i, so $e(\mathcal{B}, i)/p_i \leq s_i$ for all $i \in [n]$.

[2]A good with $b_i = 0$ for all $\boldsymbol{b} \in \mathcal{B}$ may be removed from the market.

(ii) *Budget-feasible* at prices $\boldsymbol{p}$ if every bid $\boldsymbol{b}$ spends weakly less than its budget on non-reject goods. The remaining (non-negative) amount is always spent on the reject good, so $e(\boldsymbol{b}, [n]_0) = \beta(\boldsymbol{b})$ for all $\boldsymbol{b} \in \mathcal{B}$.

(iii) *Demand-valid* at prices $\boldsymbol{p}$ if each bid only spends on a good if it demands it, so $j \in I_{\boldsymbol{b}}(\boldsymbol{p})$ if $e(\boldsymbol{b}, j) > 0$.

An expenditure is *feasible* if (i) and (ii) hold, and it is *valid* if (iii) also holds. We extend the same terminology to the allocations π corresponding to feasible or valid expenditures defined by $\pi(\boldsymbol{b}, j) = \frac{e(\boldsymbol{b},j)}{p_j}$. Note that for any valid allocation π at prices $\boldsymbol{p}$, the bundle of goods that bidder is allocated under π lies in $D_{\boldsymbol{b}}(\boldsymbol{p})$.

Definition 2. Prices $\boldsymbol{p}$ are *feasible* if there exists an expenditure e valid at $\boldsymbol{p}$.

Market objectives. We consider two separate objectives for the seller. She may set prices with the objective of maximising her revenue, or she may set prices to maximise social welfare. Intuitively (and we prove this below), at the social optimum, the seller's entire supply is allocated. To maximise revenue, the seller may prefer to raise prices and keep some of her supply.

The *revenue maximisation problem* consists of a bid list $\mathcal{B}$ and a supply vector $\boldsymbol{s} \in \mathbb{R}^n_+$. A solution to this problem is a price vector $\boldsymbol{p}$ and allocation π that maximise revenue $\sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} p_i \pi(\boldsymbol{b}, i)$ subject to π being valid at $\boldsymbol{p}$. We define the revenue function as $R(\boldsymbol{p}) := \max_\pi \sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} p_i \pi(\boldsymbol{b}, i)$ s.t. π is valid.

Recall that the buyers are price takers. Thus, social welfare (and efficiency below) is defined with respect to the submitted values. The *social welfare maximisation problem* consists of a bid list $\mathcal{B}$ and a supply vector $\boldsymbol{s} \in \mathbb{R}^n_+$. A solution to this problem is a price vector and allocation $(\boldsymbol{p}, \pi)$ that maximises social welfare $\sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i \pi(\boldsymbol{b}, i)$ subject to π being a valid allocation.

2.1 Efficiency and competitive equilibrium

We define efficiency formally and show that the first and second welfare theorem hold in our market.

Definition 3 (Efficiency)**.** A feasible allocation π is *efficient* if $\sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i \pi(\boldsymbol{b}, i) \geq \sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i \pi'(\boldsymbol{b}, i)$ for all feasible allocations $\pi' \neq \pi$.

Definition 4 (Competitive equilibrium)**.** Given a bid list $\mathcal{B}$ and a supply vector $\boldsymbol{s}$, prices $\boldsymbol{p}$ and allocation π (at $\boldsymbol{p}$) form a competitive equilibrium if, and only if, π is valid at $\boldsymbol{p}$ and $\sum_{\boldsymbol{b}\in\mathcal{B}} \pi(\boldsymbol{b}, i) = s_i$ for all goods i with $p_i > 0$.

The following technical lemma allows us to prove the first and second welfare theorem.

Lemma 1. *Suppose π is a budget-feasible allocation at prices $\boldsymbol{p}$ and fix bid $\boldsymbol{b}$. Then $\sum_{i\in[n]_0} (b_i - p_i)\pi(\boldsymbol{b}, i) \geq \sum_{i\in[n]_0} (b_i - p_i)\pi'(\boldsymbol{b}, i)$ for all budget-feasible allocations π' if, and only if, $\pi(\boldsymbol{b}) \in D_{\boldsymbol{b}}(\boldsymbol{p})$.*

Proof. Let π and π' be two allocations budget-feasible at $\boldsymbol{p}$ with corresponding expenditures e and e'. We fix a bid $\boldsymbol{b}$. Note that $e(\boldsymbol{b}, [n]_0) = e'(\boldsymbol{b}, [n]_0)$, as every bid spends its entire budget (some, or all, of it on the reject good 0). Hence,

$$
\begin{aligned}
\sum_{i\in[n]_0} (b_i - p_i)\pi(\boldsymbol{b}, i) &= \sum_{i\in I_{\boldsymbol{b}}(\boldsymbol{p})} \left(\frac{b_i}{p_i} - 1\right) e(\boldsymbol{b}, i) \\
&= \max_{i\in[n]_0} \left(\frac{b_i}{p_i} - 1\right) e(\boldsymbol{b}, [n]_0) \\
&= \max_{i\in[n]_0} \left(\frac{b_i}{p_i} - 1\right) e'(\boldsymbol{b}, [n]_0) \\
&\geq \sum_{i\in[n]_0} \left(\frac{b_i}{p_i} - 1\right) e'(\boldsymbol{b}, i) \\
&= \sum_{i\in[n]_0} (b_i - p_i)\pi'(\boldsymbol{b}, i).
\end{aligned}
$$

□

Theorem 2 (First welfare theorem)**.** *If $(\boldsymbol{p}, \pi)$ is a competitive equilibrium, π is an efficient allocation.*

Proof. Let $(\boldsymbol{p}, \pi)$ be a competitive equilibrium with corresponding expenditure e. Let π' be a feasible allocation at $\boldsymbol{p}$ with expenditure e'. Lemma 1 implies

$$\sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} (b_i - p_i)\pi(\boldsymbol{b}, i) \geq \sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} (b_i - p_i)\pi'(\boldsymbol{b}, i).$$

Rearranging and using the fact that $e(\mathcal{B}, [n]) = \boldsymbol{p}\cdot\boldsymbol{s} \geq e'(\mathcal{B}, [n])$, we have

$$\sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i\pi(\boldsymbol{b}, i) - \sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i\pi'(\boldsymbol{b}, i) \geq e(\mathcal{B}, [n]) - e'(\mathcal{B}, [n]) \geq 0.$$

□

Theorem 3 (Second welfare theorem)**.** *Let $(\boldsymbol{p}, \pi)$ be a competitive equilibrium and π' be another efficient and feasible allocation. Then $(\boldsymbol{p}, \pi')$ is also a competitive equilibrium.*

Proof. By definition of efficiency, we first note that

$$\sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i\pi(\boldsymbol{b}, i) = \sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i\pi'(\boldsymbol{b}, i). \tag{1}$$

Secondly, as π is a competitive equilibrium and π' is feasible, we have $\pi(\mathcal{B}) = \boldsymbol{s}$ and $\pi'(\mathcal{B}) \leq \boldsymbol{s}$. Multiplying both terms with $\boldsymbol{p}$, we get

$$\boldsymbol{p}\cdot\pi(\mathcal{B}) \geq \boldsymbol{p}\cdot\pi'(\mathcal{B}). \tag{2}$$

We now show that equality must hold for (2) by demonstrating that strict inequality contradicts (1). This proves our claim. Indeed, using (2) with strict inequality and Lemma 1, we obtain

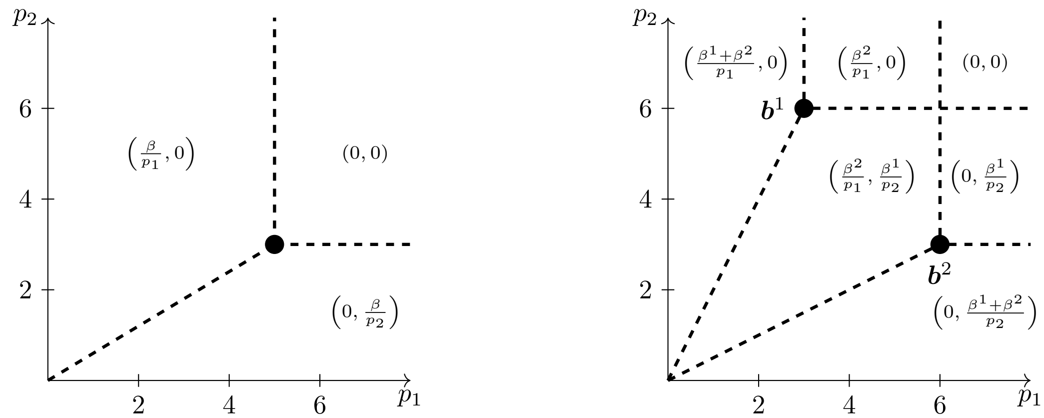
$$\begin{aligned}
\sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i\pi(\boldsymbol{b}, i) &= \sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} (b_i - p_i)\pi(\boldsymbol{b}, i) + \boldsymbol{p}\cdot\pi(\mathcal{B}) && (3)\\
&> \sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} (b_i - p_i)\pi'(\boldsymbol{b}, i) + \boldsymbol{p}\cdot\pi'(\mathcal{B}) && (4)\\
&= \sum_{\boldsymbol{b}\in\mathcal{B}, i\in[n]} b_i\pi'(\boldsymbol{b}, i). && (5)
\end{aligned}$$

□

3 Element-wise minimal prices

Recall that prices $\boldsymbol{p}$ are feasible if there exists a valid allocation π at $\boldsymbol{p}$. We show that set of feasible prices form a lower semi-lattice. In particular, there exists a special price vector $\boldsymbol{p}^*$ that is element-wise smaller than all other feasible prices (so that $\boldsymbol{p}^* \leq \boldsymbol{p}$ for all feasible $\boldsymbol{p}$). For any two price vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, we let $\boldsymbol{p}\wedge\boldsymbol{q}$ denote their *element-wise minimum* defined as $(\boldsymbol{p}\wedge\boldsymbol{q})_i = \min\{p_i, q_i\}$. The following lemma is central to our proof.

Lemma 4. *If $\boldsymbol{p}$ and $\boldsymbol{q}$ are feasible, then so is their element-wise minimum $\boldsymbol{p}\wedge\boldsymbol{q}$.*



(a) A single arctic bid.

(b) Two arctic bids with budgets β^1 and β^2.

Figure 1: The demand correspondences of arctic bids. Left: An arctic bid with budget β divides price-space into three regions (left). At prices that are element-wise larger than $\boldsymbol{b}$, the bid rejects. At prices $\boldsymbol{p}$ that lie above the line connecting $\boldsymbol{b}$ to the origin, the bid demands a quantity β/p_1 of good 1. Conversely, at prices below the line, $\boldsymbol{b}$ demands $\beta(\boldsymbol{b})/p_2$ of good 2. Right: The demand correspondence of two arctic bids $\boldsymbol{b}^1 = (3, 6; \beta^1)$ and $\boldsymbol{b}^2 = (6, 3; \beta^2)$.

Fix feasible prices $\boldsymbol{p}$ and $\boldsymbol{p}'$ with element-wise minimum $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{p}'$, and let π and π' respectively denote valid allocations at $\boldsymbol{p}$ and $\boldsymbol{p}'$. In order to prove Lemma 4, we construct an allocation τ at $\boldsymbol{r}$ and show that it is valid. We first define the set of goods A in which $\boldsymbol{p}$ is strictly dominated by $\boldsymbol{p}'$, and its complement B, so $A = \{i \in [n] \mid p_i < p'_i\}$ and $B = \{i \in [n] \mid p_i \geq p'_i\}$. Then our allocation τ at $\boldsymbol{r}$ is given by

$$\tau(\boldsymbol{b}, \cdot) = \begin{cases} \pi'(\boldsymbol{b}, \cdot) & \text{if } \boldsymbol{b} \text{ demands some good } i \in B \text{ at } \boldsymbol{r}, \\ \pi(\boldsymbol{b}, \cdot) & \text{otherwise.} \end{cases} \tag{6}$$

In order to prove the validity of τ, we first state a technical lemma that establishes the connection between the demand of a bid at $\boldsymbol{p}$, $\boldsymbol{p}'$ and $\boldsymbol{r}$.

Lemma 5. *Suppose bid $\boldsymbol{b}$ demands good $i \in A$ at $\boldsymbol{r}$. Then $\boldsymbol{b}$ also demands i at $\boldsymbol{p}$ and $I_{\boldsymbol{b}}(\boldsymbol{p}) \subseteq I_{\boldsymbol{b}}(\boldsymbol{r})$. Similarly, suppose $\boldsymbol{b}$ demands $i \in B$ at $\boldsymbol{r}$. Then it also demands i at $\boldsymbol{p}'$, and $I_{\boldsymbol{b}}(\boldsymbol{p}') \subseteq I_{\boldsymbol{b}}(\boldsymbol{r})$. Moreover, we have $I_{\boldsymbol{b}}(\boldsymbol{p}') \subseteq B$.*

Proof. Fix a bid $\boldsymbol{b}$ that demands good $i \in A$ at $\boldsymbol{r}$. As $p_i = r_i$, this implies $\frac{b_i}{p_i} = \frac{b_i}{r_i} \geq \frac{b_j}{r_j} \geq \frac{b_j}{p_j}$ for all goods $j \in [n]_0$. The first inequality holds due to the definition of demand, and the second inequality follows from $r_j \leq p_j, \forall j \in [n]_0$. Hence, $\boldsymbol{b}$ demands good i at $\boldsymbol{p}$. For the second claim that $I_{\boldsymbol{b}}(\boldsymbol{p}) \subseteq I_{\boldsymbol{b}}(\boldsymbol{r})$, fix a good $k \in I_{\boldsymbol{b}}(\boldsymbol{p})$. Then we have $\frac{b_k}{r_k} \geq \frac{b_k}{p_k} \geq \frac{b_i}{p_i} = \frac{b_i}{r_i} \geq \frac{b_j}{r_j}$ for all goods $j \in [n]_0$. The first inequality holds due to $r_k \leq p_k$, and the second and third inequalities follow from the fact that $\boldsymbol{b}$ demands k at $\boldsymbol{p}$ and i at $\boldsymbol{r}$. Hence, if bid $\boldsymbol{b}$ demands good k at $\boldsymbol{p}$, then it demands k at $\boldsymbol{r}$.

Now suppose that $\boldsymbol{b}$ demands $i \in B$ at $\boldsymbol{r}$. The proof of the first claim is identical to the case $i \in A$. We prove the last claim that $I_{\boldsymbol{b}}(\boldsymbol{p}') \subseteq B$. Suppose, for contradiction, that $\boldsymbol{b}$ demands a good $k \in A$ at $\boldsymbol{p}'$, and good $i \in B$ at $\boldsymbol{r}$. This implies $\frac{b_k}{p'_k} < \frac{b_k}{p_k} = \frac{b_k}{r_k} \leq \frac{b_i}{r_i} = \frac{b_i}{p'_i}$, in contradiction to the fact that k is demanded at $\boldsymbol{p}'$. □

We can now prove Lemma 4.

Proof of Lemma 4. Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be two feasible prices and $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{p}'$ denote their element-wise minimum. As above, π and π' are valid allocations at $\boldsymbol{p}$ and $\boldsymbol{p}'$, and τ is defined as in (6). First we see that τ is indeed an allocation, as $\tau(\boldsymbol{b}, j) \geq 0$ for all bids $\boldsymbol{b}$ and goods j. It remains to prove that τ satisfies the three criteria of Definition 1 that define validity. We can partition bids into two sets: the set $\mathcal{B}_B \subseteq \mathcal{B}$ of bids that demand a good in B at $\boldsymbol{r}$, and the set $\mathcal{B}_A = \mathcal{B} \setminus \mathcal{B}_B$ of bids that do not. Note that, by Lemma 5, the bids in $\mathcal{B}_B$ demand only goods in B at $\boldsymbol{p}'$ and, by definition, the bids in $\mathcal{B}_A$ demand only goods in A at $\boldsymbol{p}$. It follows that, under π, each bid $\boldsymbol{b} \in \mathcal{B}_A$ is only allocated quantities of goods in A (so $\pi(\mathcal{B}_A, j) > 0$ only if $j \in A$), and, similarly, every bid $\boldsymbol{b} \in \mathcal{B}_B$ only receives quantities of goods in B under π' (so $\pi(\mathcal{B}_B, j) > 0$ only if $j \in B$).

First we show that τ is supply-feasible at $\boldsymbol{r}$. Recall that τ is supply-feasible if $\tau(\mathcal{B}, j) \leq s_j$ for all $j \in [n]$. Fix some good $j \in A$. Then by definition of τ, we have $\tau(\mathcal{B}, j) = \pi(\mathcal{B}_A, j) + \pi'(\mathcal{B}_B, j)$. Recalling that $\pi'(\mathcal{B}_B, j) = 0$ and making use of the validity of π at $\boldsymbol{p}$, we get $\tau(\mathcal{B}, j) \leq \pi(\mathcal{B}, j) \leq s_j$. This immediately implies that τ satisfies part (i) of Definition 1 at $\boldsymbol{r}$ for all goods $j \in A$. Analogously, we can show that τ satisfies (i) at $\boldsymbol{r}$ for all goods $j \in B$ by recalling that $pi(\mathcal{B}_A, j) = 0$ for any good $j \in B$. As $A \cup B = [n]$, we have shown that τ is supply-feasible.

Next we argue that τ is budget-feasible at $\boldsymbol{r}$. Indeed, note that any bid $\boldsymbol{b} \in \mathcal{B}_A$ only demands goods in A. Moreover, the prices of goods in A are the same at $\boldsymbol{p}$ and $\boldsymbol{r}$, by construction of $\boldsymbol{r}$. It follows that $\boldsymbol{b}$ spends the same under τ at $\boldsymbol{r}$ as it does under π at $\boldsymbol{p}$. As π is budget-feasible at $\boldsymbol{p}$, we see that τ satisfies part (ii) of Definition 1 for all bids in $\mathcal{B}_A$. Similarly, as the bids in $\mathcal{B}_B$ only demand good in B, we apply the same argument to see that τ satisfies part (ii) for all bids in $\mathcal{B}_B$. As $\mathcal{B} = \mathcal{B}_A \cup \mathcal{B}_B$, τ is budget-feasible.

Finally, we show the demand-validity of τ at $\boldsymbol{r}$. Lemma 5 and the validity of π at $\boldsymbol{p}$ together imply that, for any $j \in [n]$, we have $\tau(\boldsymbol{b}, j) = \pi(\boldsymbol{b}, j) > 0$ only if $j \in I_{\boldsymbol{b}}(\boldsymbol{p}) \subseteq I_{\boldsymbol{b}}(\boldsymbol{r})$, which implies that τ satisfies part (iii) of Definition 1 at $\boldsymbol{r}$ for all bids in $\mathcal{B}_A$. Analogously, as $\tau(\boldsymbol{b}, j) = \pi'(\boldsymbol{b}, j)$ for all bids $\boldsymbol{b} \in \mathcal{B}_B$, we see that τ satisfies constraint (iii) at $\boldsymbol{r}$ for all bids in $\mathcal{B}_B$. Hence, τ is demand-valid at $\boldsymbol{r}$, and thus valid. □

Corollary 6. *There exists an element-wise minimal price vector* $\boldsymbol{p}^*$.

Proof. Suppose there exists no such price vector. This means that for all feasible $\boldsymbol{p}$, there exists $\boldsymbol{q}$ with $q_i < p_i$ for at least one good $i \in [n]$. Fix some feasible prices $\boldsymbol{p}$ with the property that $\boldsymbol{p}$ cannot be reduced any further in any direction. Such a point must exist, as the feasible region is closed and restricted to $\mathbb{R}^n_+$. By assumption, there exists $\boldsymbol{q}$ with $q_i < p_i$ for some $i \in [n]$. Now consider $\boldsymbol{p}' = \boldsymbol{p} \wedge \boldsymbol{q}$. By Lemma 4, $\boldsymbol{p}'$ is feasible. But as $\boldsymbol{p}' \leq \boldsymbol{p}$ with $\boldsymbol{p}'_i < \boldsymbol{p}_i$, this contradicts our assumption that $\boldsymbol{p}$ cannot be reduced further. □

4 Maximising revenue

In Section 3, we established that the set of feasible prices contain a unique element-wise minimal price vector $\boldsymbol{p}^*$. We now show that revenue is maximised at these prices. Note that we do not assume $\boldsymbol{p}^*$ to be the *only* prices at which revenue is maximised; indeed, there can be many revenue-maximising prices. However, $\boldsymbol{p}^*$ is bidder-optimal among revenue-maximising prices in the sense that it maximises the quantities of goods allocated.

Lemma 7. *For any two distinct feasible price vectors* $\boldsymbol{p}$ *and* $\boldsymbol{p}'$ *with* $\boldsymbol{p} \leq \boldsymbol{p}'$ *we have* $R(\boldsymbol{p}) \geq R(\boldsymbol{p}')$. *In other words, the maximum obtainable revenue at* $\boldsymbol{p}$ *is weakly greater than the revenue obtainable at* $\boldsymbol{p}'$.

Proof. Let π and π' be valid allocations that respectively maximise revenue at $\boldsymbol{p}$ and $\boldsymbol{p}'$. Our goal is to determine a valid allocation τ that achieves a weakly greater revenue at $\boldsymbol{p}$ than π' does at $\boldsymbol{p}'$. As $R(\boldsymbol{p}) \geq \sum_{\boldsymbol{b} \in \mathcal{B}, i \in [n]} p_i \tau(\boldsymbol{b}, i)$ and $R(\boldsymbol{p}') = \sum_{\boldsymbol{b} \in \mathcal{B}, i \in [n]} p'_i \pi'(\boldsymbol{b}, i)$, this immediately implies the result.

If $\boldsymbol{p} = \boldsymbol{p}'$, there is nothing to prove. Hence we assume that $S := \{i \in [n] \mid p_i < p'_i\}$, the set of goods which are priced strictly lower at $\boldsymbol{p}$ than at $\boldsymbol{p}'$, is non-empty. Fix a bid $\boldsymbol{b} \in \mathcal{B}$. In order to define the new allocation $\tau(\boldsymbol{b}, \cdot)$ to $\boldsymbol{b}$ at $\boldsymbol{p}$, we distinguish between the two cases that $I_{\boldsymbol{b}}(\boldsymbol{p})$ is, and is not, a subset of S.

Case 1: Suppose $\boldsymbol{b}$ demands a subset of S at $\boldsymbol{p}$, so $I_{\boldsymbol{b}}(\boldsymbol{p}) \subseteq S$. In this case, we set $\tau(\boldsymbol{b}, \cdot) = \pi(\boldsymbol{b}, \cdot)$. As π is valid and τ and π are both allocations at the same prices, the bid $\boldsymbol{b}$ spends its entire budget under τ. Moreover, it is only allocated goods that it demands.

Case 2: Suppose $I_{\boldsymbol{b}}(\boldsymbol{p}) \not\subseteq S$. We note that $I_{\boldsymbol{b}}(\boldsymbol{p}') \cap S = \emptyset$ and $\boldsymbol{b}$ still demands all goods in $I_{\boldsymbol{b}}(\boldsymbol{p}')$, so $I_{\boldsymbol{b}}(\boldsymbol{p}') \subseteq I_{\boldsymbol{b}}(\boldsymbol{p})$. In this case, we set $\tau(\boldsymbol{b}, \cdot) = \pi'(\boldsymbol{b}, \cdot)$. As $\boldsymbol{b}$ is only allocated goods not in S, and $p_i = p'_i$ for all goods $i \in [n]_0 \setminus S$, it follows that $\boldsymbol{b}$ spends the same at both prices.

Note that, in both cases, $\boldsymbol{b}$ is only allocated goods that it demands. To summarise, we define τ as

$$\tau(\boldsymbol{b}, \cdot) = \begin{cases} \pi(\boldsymbol{b}, \cdot) & \text{if } \boldsymbol{b} \text{ demands a subset of } S \text{ at } \boldsymbol{p}, \\ \pi'(\boldsymbol{b}, \cdot) & \text{otherwise.} \end{cases}$$

We now prove that τ is valid. We have already argued above that all bids satisfy the demand and budget conditions of Definition 1. It remains to show that aggregate demand does not exceed supply s_i for any goods $i \in [n]_0$ under τ. Note that a bid is allocated a good i in S under τ if, and only if, it satisfies Case 1 above. Indeed, in this case we set $\tau(\boldsymbol{b}, i) = \pi(\boldsymbol{b}, i)$. Hence, for any $i \in S$, we have $\sum_{\boldsymbol{b} \in \mathcal{B}} \tau(\boldsymbol{b}, i) \leq \sum_{\boldsymbol{b} \in \mathcal{B}} \pi(\boldsymbol{b}, i) \leq s_i$. Similarly, for any $i \notin S$ the bid will satisfy Case 2, and get $\sum_{\boldsymbol{b} \in \mathcal{B}} \tau(\boldsymbol{b}, i) \leq \sum_{\boldsymbol{b} \in \mathcal{B}} \pi'(\boldsymbol{b}, i) \leq s_i$.

Finally, we see that τ achieves weakly greater revenue at $\boldsymbol{p}$ than π' does at $\boldsymbol{p}'$. To see this, note that each bid satisfying Case 1 spends its entire budget on non-reject goods and thus contributes a weakly greater amount to revenue at $\boldsymbol{p}$ than at $\boldsymbol{p}'$, while a bid satisfying Case 2 contributes the same amount to revenue. □

Corollary 8. *Revenue is maximised at element-wise minimal feasible prices* $\boldsymbol{p}^*$.

Proof. Suppose $\boldsymbol{p} \neq \boldsymbol{p}^*$ is a revenue-maximising price vector. As $\boldsymbol{p}^*$ is element-wise minimal, we have $\boldsymbol{p}^* \leq \boldsymbol{p}$. Then by Lemma 7, we can obtain weakly greater revenue at $\boldsymbol{p}^*$ than at $\boldsymbol{p}$, or $R(\boldsymbol{p}^*) \geq R(\boldsymbol{p})$. But as $\boldsymbol{p}$ maximises R, so does $\boldsymbol{p}^*$. □

5 Market-clearing prices are unique

Theorem 9. *There is a unique price vector* $\boldsymbol{p}^*$ *at which the market is cleared.*

Proof. Suppose we have a valid allocation π that exhausts supply $\boldsymbol{s}$. We want to find the corresponding prices $\boldsymbol{p}$ that support π. First, we formulate the problem of finding $\boldsymbol{p} \geq 0$ as a system of linear inequalities by writing out the budget constraints as equalities and the demand constraints as inequalities. The latter is achieved by adding, for every bid $\boldsymbol{b}$ that demands a positive amount of good i, the inequalities $\frac{b_i}{p_i} \geq \frac{b_j}{p_j}, \forall j \in [n]_0$. Our goal is to prove that there is exactly one solution to this system of inequalities. In Section 6, we show that the element-wise minimal prices $\boldsymbol{p}^*$ clear the market, so the polytope defined by this system has at least one solution.[3]

For contradiction, suppose there is a second feasible price vector $\boldsymbol{p} \geq \boldsymbol{p}^*$ that also clears the market. Then by convexity of the polytope, it follows that $\boldsymbol{p}' = \boldsymbol{p}^* + \varepsilon(\boldsymbol{p} - \boldsymbol{p}^*)$ is also market-clearing for any $0 \leq \varepsilon \leq 1$. From now on, we assume that $\boldsymbol{p}'$ is obtained by letting ε be infinitesimally small. Let $S := \{i \in [n] \mid p_i^* < p_i\}$ denote the prices that change when we move from $\boldsymbol{p}^*$ to $\boldsymbol{p}'$. As $\boldsymbol{p} \neq \boldsymbol{p}^*$, S is non-empty. We first make a technical observation about the demand of each bid at $\boldsymbol{p}^*$ and $\boldsymbol{p}'$.

Observation 1. For any bid $\boldsymbol{b} \in \mathcal{B}$, we note: (i) If $I_{\boldsymbol{b}}(\boldsymbol{p}^*) \not\subseteq S$, then $I_{\boldsymbol{b}}(\boldsymbol{p}') = I_{\boldsymbol{b}}(\boldsymbol{p}^*) \setminus S$. (ii) If $I_{\boldsymbol{b}}(\boldsymbol{p}^*) \subseteq S$, then $I_{\boldsymbol{b}}(\boldsymbol{p}') \subseteq I_{\boldsymbol{b}}(\boldsymbol{p}^*)$. This holds because ε was chosen to be infinitesimally small.

Now suppose e' is a valid expenditure at $\boldsymbol{p}'$ that clears the market. Let $\mathcal{B}_S := \{b \in \mathcal{B} \mid I_{\boldsymbol{b}}(\boldsymbol{p}^*) \subseteq S\}$ denote the set of bids satisfying case (ii) of Observation 1. We take a look at the revenue contributions from each bid towards goods in S. At $\boldsymbol{p}'$, Observation 1 tells us that only bids $\boldsymbol{b} \in \mathcal{B}_S$ spend on goods in S, and they each spend their entire budget on S. Hence, the budget and supply feasibility of e' imply $\beta(\mathcal{B}_S) = e'(\mathcal{B}_S, S) = e'(\mathcal{B}, S) = \sum_{i \in S} p'_i s_i$. At $\boldsymbol{p}^*$, the bids of $\mathcal{B}_S$ also spend their entire budget on S, but it may be the case that other bids also spend part of their budget on S. Hence, $\beta(\mathcal{B}_S) = e(\mathcal{B}_S, S) \leq e(\mathcal{B}, S) = \sum_{i \in S} p_i^* s_i$. It follows that $\sum_{i \in S} p'_i s_i \leq \sum_{i \in S} p_i^* s_i$, in contradiction to the fact that $p'_i > p_i^*$ for all $i \in S$. □

[3]Indeed, the algorithm in Section 7 finds these market-clearing prices.

Corollary 10. *The element-wise minimal prices $\boldsymbol{p}^*$ uniquely induce an efficient allocation and a competitive equilibrium.*

Proof. This is immediate from the welfare theorems and Theorem 9. □

6 The element-wise minimal feasible prices clear the market

Suppose $\boldsymbol{p}$ is a feasible price vector at which no allocation exhausts supply $\boldsymbol{s}$ in all goods. In this case, we present a method to scale a subset of goods uniformly in price by a factor $0 < c < 1$ while retaining feasibility and increasing aggregate demand. As stated in Corollary 12, this implies our main result that the element-wise minimal prices $\boldsymbol{p}^*$ clear the market.

In order to describe this method of scaling prices, we first introduce the demand and expenditure graphs. Then we present a subroutine that – given prices $\boldsymbol{p}$, an expenditure e and an unsated good i under e at $\boldsymbol{p}$ – either returns scaled-down prices or an expenditure at $\boldsymbol{p}$ under which i is sated. After invoking this subroutine $O(n)$ times, prices will be scaled down. We note that the subroutine defined here is also called in our algorithm (cf. Section 7) to find the unique market-clearing prices $\boldsymbol{p}^*$.

6.1 The demand and expenditure graphs

In order to describe demand and spending relationships between bids and goods, we introduce two bipartite graphs, the *demand graph* and the *expenditure graph*. Modulating expenditures in appropriately defined sub-trees of these graphs is the central component in our price reduction procedure.

Definition 5 (Demand graph and expenditure graph)**.** Let e be a valid expenditure at feasible prices $\boldsymbol{p}$. We denote by $\mathcal{D}$ the bipartite *demand graph* on vertex sets $\mathcal{B}$ and $[n]_0$. There is an edge $(\boldsymbol{b}, i) \in D$ if, and only if, $\boldsymbol{b}$ demands i at $\boldsymbol{p}$. Moreover, we define the *expenditure graph* $\mathcal{E}$ of $\mathcal{D}$ induced by the edges $(\boldsymbol{b}, i)$ with positive expenditure $e(\boldsymbol{b}, i) > 0$. Finally, $\mathcal{D}_i$ and $\mathcal{E}_i$ respectively denote the connected component of $\mathcal{D}$ and $\mathcal{E}$ that contains good i.

Note that $\mathcal{E}_i$ is a subgraph of $\mathcal{D}_i$ for all goods $i \in [n]_0$, and each $\mathcal{D}_i$ can contain multiple connected components of $\mathcal{E}$. Both graphs are illustrated in Fig. 2.

Breaking cycles. Our price reduction method assumes that the demand graph is acyclic. For this reason, we follow [14] in perturbing bids by an infinitesimal amount in order to break cycles. Induce an ordering $\mathcal{B} = \{\boldsymbol{b}^1, \boldsymbol{b}^2, \ldots\}$ on the bids, and perturb each entry b_i^k by adding $\varepsilon^{kn} + \varepsilon^i$ for some infinitesimally small $\varepsilon > 0$. In our algorithms, this perturbation can be simulated without a running-time penalty by implementing lexicographic tie-breaking when constructing the demand and expenditure graphs. For details about the perturbation, we refer to [14].

6.2 Reducing prices along a tree

We now describe the subroutine introduced above. The full procedure is stated in Algorithm 1 and elucidated below. Correctness is proved in Theorem 11.

Identifying goods to reduce in price. Suppose i is an unsated good under expenditure e at $\boldsymbol{p}$. We describe a procedure which either identifies a set J of goods to reduce in price, or finds a new expenditure e' at $\boldsymbol{p}$ under which good i is sated. We first construct a directed tree T by rooting the (undirected) tree $\mathcal{D}_i$ at good i and then remove all subtrees rooted at endpoint goods j of arcs $(\boldsymbol{b}, j) \in \mathcal{D}_i \setminus \mathcal{E}$. This is illustrated in Fig. 2. For notational convenience, we also let T_v denote the subtree of T rooted at v for any vertex $v \in T$.

If T does not contain the reject good 0, we let J consist of the goods of T and are done. Otherwise, we consider the directed path $(i = g^1, \boldsymbol{b}^1, \ldots, g^k, \boldsymbol{b}^k, g^{k+1} = 0)$ from i to 0 in T and modify the expenditure e to redirect spending from 0 to i as follows. Let $m > 0$ be the minimum of $p_i s_i - e(\mathcal{B}, i)$ and $\min_{j\in[k]} e(\boldsymbol{b}^j, g^j)$.

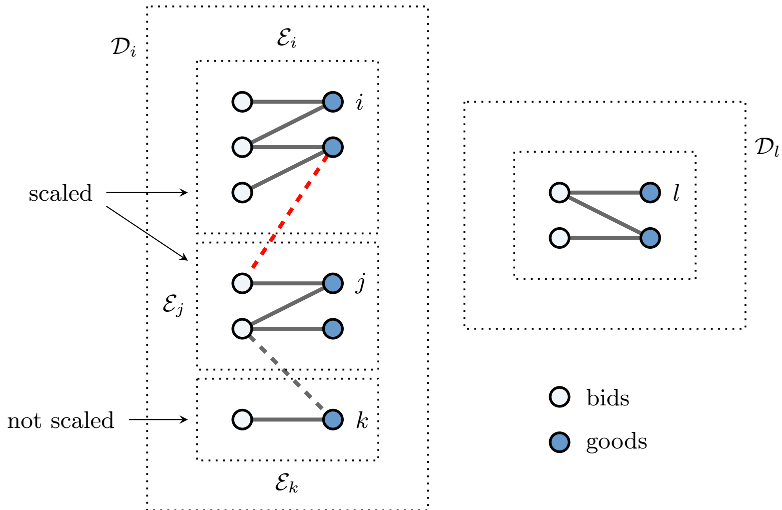


Figure 2: An illustration of the demand and expenditure graphs. Solid and dashed lines respectively indicate edges in $\mathcal{E}$ and $\mathcal{D} \setminus \mathcal{E}$. The component $\mathcal{D}_i$ contains three components of $\mathcal{E}$. If we root $\mathcal{D}_i$ at i, we note that there is an arc (dashed red) from a good in $\mathcal{E}_i$ to a bid in $\mathcal{E}_j$, and an arc (dashed grey) from a bid in $\mathcal{E}_j$ to a good in $\mathcal{E}_k$. Hence the tree constructed in Section 6.2, which contains all goods to be scaled down in price, consists of $\mathcal{E}_i$ and $\mathcal{E}_j$ but not $\mathcal{E}_k$.

Algorithm 1 Tree-based price reduction

Input: Feasible prices $\boldsymbol{p}$, valid expenditure e and unsated good i.

Output: Prices $\boldsymbol{p}' \leq \boldsymbol{p}$ and expenditure e', where either $\boldsymbol{p}' = \boldsymbol{p}$ and i is sated, or $p'_j < p_j$ is for some goods $j \in [n]$.

1: Construct tree T by rooting $\mathcal{D}_i$ at i and removing subtrees rooted at goods j of arcs $(\boldsymbol{b}, j) \in \mathcal{D}_i \setminus \mathcal{E}$.
2: **if** $0 \notin T$ **then** go to step 9. **end if**
3: Compute the path $i = g^1, \boldsymbol{b}^1, \ldots, g^k, \boldsymbol{b}^k, g^{k+1} = 0$ from i to 0 in T and the minimum m of $p_i s_i - e(\mathcal{B}, i)$ and $\min_{j \in [k]} e(g^j, \boldsymbol{b}^j)$. Update e by $e(\boldsymbol{b}^j, g^j) \mathrel{+}= m$ and $e(\boldsymbol{b}^j, g^{j+1}) \mathrel{-}= m$.
4: **if** good i is now sated under e **then**
5:     **return** $\boldsymbol{p}$ and e.
6: **else**
7:     Let j be the smallest such value $j \in [k]$ and remove from T the subtree rooted at g^{j+1}.
8: **end if**
9: Traverse T depth-first to recursively compute $\sum_{k \in T_v} \alpha_k p_k s_k$ for each vertex $v \in T$.
10: Compute $c^{(1)}, c^{(2)}$ and $c^{(3)}$ as specified in Section 6.2, and scaling factor $c^* = \max\left\{c^{(1)}, c^{(2)}, c^{(3)}\right\}$.
11: Traverse T depth-first to recursively compute $\Delta(\boldsymbol{b}, j)$ for each arc in T. Compute e' as described in (7).
12: Let $p'_j = c^* p_j$ for all $j \in T$ and $p'_j = p_j$ otherwise.
13: **return** $\boldsymbol{p}'$ and e'.

We let $e'(\boldsymbol{b}^j, g^j) = e(\boldsymbol{b}^j, g^j) + m$ and $e'(\boldsymbol{b}^j, g^{j+1}) = e(\boldsymbol{b}^j, g^j) - m$ for all $j \in [k]$. Note that the resulting expenditure is valid and redirecting spending in this way does not add edges to the expenditure graph, or cause the expenditure on any goods in $[n]$ to reduce. In particular, no sated good becomes unsated.

If $m = p_i s_i - e(\mathcal{B}, i)$, then good i is sated under the new expenditure e', and we are done. Otherwise, if $m < p_i s_i - e(\mathcal{B}, i)$, we obtain the tree corresponding to the new expenditure e' by identifying the smallest $j \in [k]$ such that $e'(\boldsymbol{b}^j, g^{j+1}) = 0$ and removing the subtree rooted at g^{j+1} from T. As the tree no longer contains the reject good 0, we let J denote the goods in T and are done.

Scaling prices. Now we suppose that we have identified the set of goods J of the tree T not containing 0, and good i is unsated. In order to fully specify our scaling method, it remains to determine the factor $0 < c < 1$ by which to uniformly scale the prices of goods in J so that the resulting prices remain feasible. Let $\boldsymbol{p}'$ denote the prices after scaling, so $p'_j = cp_j$ for $j \in J$ and $p'_j = p_j$ otherwise. We proceed by describing an expenditure e' at $\boldsymbol{p}'$ that is derived from e by adding, or removing, a difference term parameterised by c to, or from, each $e(\boldsymbol{b}, j)$. The scaling factor c is then chosen to ensure that e' is valid. Recall that e' needs to satisfy the conditions in Definition 1 to be a valid expenditure at prices $\boldsymbol{p}'$. We recall, for instance, that the supply constraint imposes $e'(\mathcal{B}, j) \leq p'_j s_j$ for all goods $j \in [n]$. In the following, we will impose a stronger supply constraint for all goods $j \in [n] \setminus \{i\}$. Suppose that the proportion of good j aggregately demanded under e is given by $\alpha_j \in [0, 1]$, so that $e(\mathcal{B}, j) = \alpha_j p_j s_j$ for good j. Then we require that $e'(\mathcal{B}, j) = \alpha_j p'_j s_j$ for all $j \in [n] \setminus \{i\}$. (So $e'(\mathcal{B}, j) = c\alpha_j p_j s_j$ for $j \in J$ and $e'(\mathcal{B}, j) = \alpha_j p_j s_j$ for $j \notin J$.) Note that this trivially implies the supply constraint, as $\alpha_j \leq 1$ and $c < 1$.

The stronger supply condition uniquely determines an expenditure e' for any given c, which can be expressed by adding, or subtracting, the differential terms $\Delta(\boldsymbol{b}, j)$ to, or from, $e(\boldsymbol{b}, j)$. For every arc in T between some bid $\boldsymbol{b}$ and good j with endpoint $v \in \{\boldsymbol{b}, j\}$, we define $\Delta(\boldsymbol{b}, j) := (1 - c) \sum_{k \in T_v} \alpha_k p_k s_k$. For all other $\boldsymbol{b}$ and j, we let $\Delta(\boldsymbol{b}, j) = 0$. Finally, we define the new expenditure e' as

$$e'(\boldsymbol{b}, j) = \begin{cases} e(\boldsymbol{b}, j) - \Delta(\boldsymbol{b}, j), & \text{if } \boldsymbol{b} \text{ precedes } j \text{ in } T, \\ e(\boldsymbol{b}, j) + \Delta(\boldsymbol{b}, j), & \text{otherwise.} \end{cases} \tag{7}$$

It remains to find the smallest $0 < c < 1$ such that e' remains valid. Conceptually, we choose c by reducing from 1 until one of three scenarios occurs. For each scenario i, we determine the scaling factor $c^{(i)}$ at which this scenario occurs. Our scaling factor is then chosen by computing $c^* = \max\{c^{(1)}, c^{(2)}, c^{(3)}\}$. The scenarios are as follows.

1. Spending $e(\mathcal{B}, i)$ on good i increases until it becomes sated. This strictly reduces the number of unsated goods in $\mathcal{D}$.
2. Some $e'(\boldsymbol{b}, j)$ decreases to 0. This breaks up a component in $\mathcal{E}$.
3. Some bid not in T starts demanding a good in T (along with goods outside T). This connects two components in $\mathcal{D}$.

Scenario 1: We note that c must be chosen such that $e'(\mathcal{B}, i) = e(\mathcal{B}, i) + \Delta(\mathcal{B}, i) \leq cp_i s_i$. Substituting $\Delta(\mathcal{B}, i) = \sum_{k \in J \setminus \{i\}} (1 - c)\alpha_k p_k s_k$ and solving for c, we see that this holds for scaling factor

$$c^{(1)} = \frac{e(\mathcal{B}, i) + \sum_{k \in J \setminus \{i\}} \alpha_k p_k s_k}{p_i s_i + \sum_{k \in J \setminus \{i\}} \alpha_k p_k s_k}. \tag{8}$$

As $0 \leq e(\mathcal{B}, i) \leq p_i s_i$, factor $c^{(1)}$ is well-defined and $0 < c^{(1)} < 1$.

Scenario 2: Note that $\Delta(\boldsymbol{b}, j) > 0$ for any scaling factor c that is strictly less than 1. Hence, $e'(\boldsymbol{b}, j)$ is non-negative for all arcs from a good j to a bid $\boldsymbol{b}$, for any $c < 1$. Now fix an arc from bid $\boldsymbol{b}$ to good j. In this case, we need to ensure that $e(\boldsymbol{b}, j) \geq \Delta(\boldsymbol{b}, j)$ to guarantee $e'(\boldsymbol{b}, j) \geq 0$. Substituting $\Delta(\boldsymbol{b}, j) = (1 - c) \sum_{k \in T_j} \alpha_k p_k s_k$, and solving for c, we get

$$c \geq 1 - \frac{e(\boldsymbol{b}, j)}{\sum_{k \in T_j} \alpha_k p_k s_k}. \tag{9}$$

Hence $c^{(2)}$ as chosen below guarantees $e'(\boldsymbol{b}, j) \geq 0$ for every bid $\boldsymbol{b}$ and good j with equality for at least one pair $(\boldsymbol{b}, j)$.

$$c^{(2)} = \max_{(\boldsymbol{b}, j) \in T} \left(1 - \frac{e(\boldsymbol{b}, j)}{\sum_{k \in T_j} \alpha_k p_k s_k} \right). \tag{10}$$

Note that $0 < e(\boldsymbol{b}, j) \leq \alpha_j p_j s_j \leq \sum_{k \in T_j} \alpha_k p_k s_k$ for all $(\boldsymbol{b}, j) \in T$, so $c^{(2)}$ is well-defined and satisfies $0 < c^{(2)} < 1$.

Scenario 3: Fix a bid $\boldsymbol{b}$. Then $\boldsymbol{b}$ continues to demand goods outside T only if $\frac{1}{c} \max_{j \in T} \frac{b_j}{p_j} \leq \max_{k \notin T} \frac{b_k}{p_k}$. Hence, by solving for c and taking the maximum over all bids not in T, we get

$$c^{(3)} = \max_{\boldsymbol{b} \notin T} \left[\max_{j \in T} \frac{b_j}{p_j} \min_{k \notin T} \frac{p_k}{b_k} \right]. \tag{11}$$

Note that $1 \leq \max_{j \in T} \frac{b_j}{p_j} < \max_{k \in [n]_0} \frac{b_k}{p_k}$ for all $\boldsymbol{b} \notin T$, as these bids do not demand any goods in T at $\boldsymbol{p}$. Hence, $c^{(3)}$ is well-defined and $0 < c^{(3)} < 1$.

Theorem 11. *Suppose $\boldsymbol{p}$ is not market-clearing, and e is a valid expenditure at $\boldsymbol{p}$ with an unsated good i. The price-reduction procedure above (Algorithm 1), when given $\boldsymbol{p}$, e and i, either returns a valid expenditure e' at $\boldsymbol{p}$ that sates good i, or it returns reduced prices $\boldsymbol{p}'$ and a valid expenditure e' at $\boldsymbol{p}'$, in polynomial time.*

Proof. Suppose first that the subroutine terminates in line 5. Then the resulting expenditure e' that it returns is trivially valid, and we are done.

Now suppose that the subroutine does not terminate in line 5. Then we note that $\boldsymbol{p}'$ returned by the procedure is strictly smaller than $\boldsymbol{p}$ for all goods $j \in J$, as the scaling factor is strictly less than 1. Hence it suffices to show that the expenditure e' returned by Algorithm 1 after scaling prices of goods J by factor c^* is valid in the sense of Definition 1. By construction of c^*, $e'(\boldsymbol{b}, j)$ is non-negative for all bids $\boldsymbol{b}$ and goods j. We also note that all arcs in T coincide with edges in the demand graph by construction, and if $e(\boldsymbol{b}, j)$ changes, then there is an edge between $\boldsymbol{b}$ and j in the demand graph. Meanwhile, the choice of $c^{(3)}$ guarantees that all bids $\boldsymbol{b} \notin T$ demand, at $\boldsymbol{p}'$, a superset of the goods they demand at $\boldsymbol{p}$. It follows e is demand-valid.

Next, we verify that e is budget-feasible. Suppose first that $\boldsymbol{b}$ is a bid not in T. Then, by construction of T, none of the prices of goods that $\boldsymbol{b}$ demands are scaled, and the spending of $\boldsymbol{b}$ is unchanged. Next suppose that bid $\boldsymbol{b}$ is a leaf in T, and good j is its parent. Then $\Delta(\boldsymbol{b}, j) = \sum_{k \in T_{\boldsymbol{b}}} \alpha_k p_k s_k = 0$, as the tree $T_{\boldsymbol{b}}$ contains no goods. Hence, $e'(\boldsymbol{b}, j) = e(\boldsymbol{b}, j) = \beta(\boldsymbol{b})$ and the condition for budget feasibility holds. Finally, suppose bid $\boldsymbol{b}$ is not a leaf in T. Let g^1 be its parent and $g^2, \ldots, g^k$ be its children. We need to verify that $e'(\boldsymbol{b}, [n]_0) = \beta(\boldsymbol{b})$. As $e(\boldsymbol{b}, [n]_0) = \beta(\boldsymbol{b})$, it suffices to see that $-\Delta(\boldsymbol{b}, g^1) = \sum_{j=2}^{k} \Delta(\boldsymbol{b}, g^j)$. But this follows from the definition of Δ.

Next, we show that e is supply-feasible. For the originally unsated good i, this follows immediately from the choice of our scaling factor $c^* \geq c^{(1)}$ in (8). For all other goods, we verify the stronger supply condition introduced above. Suppose first that j is a good not in T. Then by construction of T, the amount that each bid spends on j remains unchanged. Next suppose good j is a leaf in T, and bid $\boldsymbol{b}$ is its parent. Then the stronger supply constraint requires that $e'(\boldsymbol{b}, j) = ce(\boldsymbol{b}, j) = c\alpha_j p_j s_j$, which agrees with our definition of $\Delta(\boldsymbol{b}, j) = (1-c)\alpha_j p_j s_j$. Finally, suppose good j is not a leaf in T, and let $\boldsymbol{b}$ be its parent and $\mathcal{B}'$ be its children. Then the stronger supply constraint is satisfied, as

$$e'(\mathcal{B}, j) = e(\mathcal{B}, j) - \Delta(\boldsymbol{b}, j) + \Delta(\mathcal{B}', j) = \alpha_j p_j s_j - (1-c)\alpha_j p_j s_j = c\alpha_j p_j s_j.$$

To see that the subroutine runs in polynomial time, note that the running time is dominated by traversing the tree with $n + |\mathcal{B}|$ nodes depth-first, which is polynomial in n and $|\mathcal{B}|$. □

6.3 The procedure for reducing prices

Recall that, given prices $\boldsymbol{p}$ that are not market-clearing, and an expenditure e, the subroutine described above either modifies e in order to ensure that an unsated good becomes sated, or reduces prices of some goods. We claim that after $O(n)$ repetitions of this subroutine with a new unsated good, prices of some goods are reduced. Indeed, we start with at most n unsated goods. Each application of the subroutine that fails to reduce prices strictly reduces the number of unsated goods, as no sated goods can revert to being unsated. In particular, this implies that the element-wise minimal prices $\boldsymbol{p}^*$ clear the market.

Algorithm 2 Finding element-wise minimal feasible prices $\boldsymbol{p}^*$

Input: List of bids $\mathcal{B}$ in general position.
Output: Element-wise smallest feasible prices $\boldsymbol{p}'$.

```
Let p be the element-wise maximum over the bids in B and e be the expenditure at which all bids spend
their entire budget on the reject good 0.
while there are unsated goods do
    Identify an unsated good i under e at p.
    while good i is unsated do
        Apply tree-based subroutine stated in Algorithm 1 with good i to update (p, e).
    end while
end while
return prices p and expenditure e.
```

Corollary 12. *Competitive equilibrium is uniquely achieved at element-wise minimal prices* $\boldsymbol{p}^*$*.*

Proof. In Section 5, we prove that market-clearing prices are unique in our market. Suppose that $\boldsymbol{p}^*$ is not market-clearing. Then by Theorem 11, and our discussion above, we can reduce prices of some goods. But this contradicts our assumption that $\boldsymbol{p}^*$ is the element-wise minimal feasible price vector. $\square$

7 Finding market-clearing prices

Here we present an algorithm to find market-clearing prices, as well as a running time analysis that shows the algorithm terminates in exponential time. The algorithm uses the tree-based price scaling subroutine introduced in Section 6. Note that each call to Algorithm 1 makes progress in the sense that prices decrease or a good is sated.

For a given bid list $\mathcal{B}$, let $\boldsymbol{p} = \bigvee_{\boldsymbol{b}\in\mathcal{B}} \boldsymbol{b}$ be the element-wise maximum over all bids. Then all bids demand the reject good, while possibly being indifferent between 0 and other goods. We define the expenditure e at $\boldsymbol{p}$ under which all bids spend their entire budget on the reject good, so for each bid $\boldsymbol{b} \in \mathcal{B}$ we let $e(\boldsymbol{b}, 0) = \beta(\boldsymbol{b})$ and $e(\boldsymbol{b}, i) = 0$ for all $i \in [n]$. This expenditure is trivially valid.

Of course, no market clears at this initial allocation. We then pick any unsated good i and call the subroutine of Algorithm 1 with good i at $(\boldsymbol{p}, e)$ to reduce prices until the market for good i clears (good i becomes sated). Subsequently, we select another unsated good, reduce prices by repeatedly calling Algorithm 1, and repeat this step until all goods are sated.

Theorem 13. *Algorithm 2 finds the market-clearing prices* $\boldsymbol{p}^*$ *in exponential time.*

Proof. We argue that the number of calls to Algorithm 1 is bounded. Note that the outer loop in Algorithm 2 iterates $O(n)$ times. Indeed, invoking Algorithm 1 does not cause any sated goods to become unsated. Every iteration of the outer loop reduces the number of unsated goods by one, and we start with at most n unsated goods.

We now show that the inner loop of Algorithm 2 applies Algorithm 1 an exponential number of times before good i is sated. Suppose, for contradiction, that the subroutine is called more often. In this case, we note that on each iteration, the scaling factor is chosen such that scenario 2, scenario 3 or both scenarios occur. Let $\mathcal{D}_i^{(k)}$ denote the connected component of $\mathcal{D}$ containing good i when scenario 2 occurs for the k-th time. In particular, we note that any two consecutive occurrences of scenario 2 can be separated by at most $|\mathcal{B}|$ iterations of scenario 3's. To see this, realise that every time only scenario 3 occurs, T grows through the addition of a bid, and we have $|\mathcal{B}|$ in total.

We now argue that all $\mathcal{D}_i^{(l)}$ are distinct. As there are exponentially many configurations of $\mathcal{D}_i$, this implies the result. Note that whenever the k-th scenario 2 occurs, we have $e'(\boldsymbol{b}, j) = 0$ for some arc $(\boldsymbol{b}, j)$, and good $j \in T$ is not in T in the next iteration. Hence, the price of good j is not scaled down in the next iteration. Meanwhile, the price of good i is scaled down on every iteration of Algorithm 1. Now suppose, for any k, that $\mathcal{D}_i^{(k)}$ reoccurs at a later stage. This implies that p_i and p_j have been scaled down the same; indeed, this follows by considering the path from i to j in $\mathcal{D}_i^{(k)}$ and applying the conditions of demand. But we have just argued that p_i and p_j have not been scaled down the same, a contradiction.

Finally, we see that the algorithm finds market-clearing prices. To see this, note that it terminates only once all goods are sated. This concludes the proof. □

8 Conclusion

We analyse a market for multiple, divisible goods, in which a unique set of prices exists that induce a socially optimal and revenue-optimal allocation. This co-existence of revenue-optimality and efficiency makes our market compelling in theory as well as highly attractive for sellers, buyers, and market platforms in practice. We provide algorithmic results to derive the efficient and optimal price vector, and expect to improve on these results in future work.

References

[1] B. Adsul, C. S. Babu, J. Garg, R. Mehta, and M. Sohoni. A simplex-like algorithm for linear Fisher markets. *Current Science*, 103(9):1033–1042, 2012.

[2] K. J. Arrow and G. Debreu. Existence of an Equilibrium for a Competitive Economy. *Econometrica*, 22(3):265–290, 1954.

[3] E. Baldwin, P. W. Goldberg, P. Klemperer, and E. Lock. Solving Strong-Substitutes Product-Mix Auctions. arXiv:1909.07313, 2019.

[4] W. C. Brainard and H. E. Scarf. How to Compute Equilibrium Prices in 1891. *American Journal of Economics and Sociology*, 64(1):57–83, 2005.

[5] N. R. Devanur and V. V. Vazirani. The spending constraint model for market equilibrium: Algorithmic, existence and uniqueness results. In *Conference Proceedings of the Annual ACM Symposium on Theory of Computing*, STOC '04, pages 519–528, Chicago, Illinois, USA, 2004.

[6] N. R. Devanur, C. H. Papadimitriou, A. Saberi, and V. V. Vazirani. Market Equilibrium via a Primal-Dual Algorithm for a Convex Program. *Journal of the ACM*, 55(5):1–18, 2008.

[7] B. C. Eaves. A finite algorithm for the linear exchange model. *Journal of Mathematical Economics*, 3 (2):197–203, 1976.

[8] M. Fichtl. Computing Candidate Prices in Budget-Constrained Product-Mix Auctions. Working paper. https://dss.in.tum.de/files/staff-files/fichtl/BudgetConstrainedPMA.pdf, accessed 12/07/2021.

[9] J. Garg, R. Mehta, M. Sohoni, and N. K. Vishnoi. Towards Polynomial Simplex-Like Algorithms for Market Equilibria. In *Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '13, pages 1226–1242, 2013.

[10] K. Jain. A polynomial time algorithm for computing an Arrow-Debreu market equilibrium for linear utilities. *SIAM Journal on Computing*, 37(1):303–318, 2007.

[11] P. Klemperer. A New Auction for Substitutes: Central Bank Liquidity Auctions, the U.S. TARP, and Variable Product-Mix Auctions. Working Paper, 2008.

[12] P. Klemperer. The Product-Mix Auction: a New Auction Design for Differentiated Goods. *Journal of the European Economic Association*, (8):526–36, 2010.

[13] P. Klemperer. Product-Mix Auctions. Nuffield College Working Paper 2018-W07, http://www.nuffield.ox.ac.uk/users/klemperer/productmix.pdf, 2018.

[14] J. B. Orlin. Improved Algorithms for Computing Fisher's Market Clearing Prices. In *Proceedings of the Annual ACM Symposium on Theory of Computing*, STOC '10, pages 291–300, Cambridge, Massachusetts, USA, 2010.

[15] L. Walras. Éléments d'économie politique pure ou théorie de la richesse sociale (elements of pure economics, or the theory of social wealth). Lausanne, Paris, 1874. (1899, 4th ed.; 1926, rev ed., 1954, Engl. transl.), 1874.