Substitutes markets with budget constraints: solving for competitive *and* optimal prices*

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Abstract. Markets with multiple divisible goods have been studied widely from the perspective of revenue and welfare. In general, it is well known that envy-free revenue-maximal outcomes can result in lower welfare than competitive equilibrium outcomes. We study a market in which buyers have quasilinear utilities with linear substitutes valuations and budget constraints, and the seller must find prices and an envy-free allocation that maximise revenue or welfare. Our setup mirrors markets such as ad auctions and auctions for the exchange of financial assets. We prove that the unique competitive equilibrium prices are also envy-free revenue-maximal. This coincidence of maximal revenue and welfare is surprising and breaks down even when buyers have piecewise-linear valuations. We present a novel characterisation of the set of 'feasible' prices at which demand does not exceed supply, show that this set has an elementwise minimal price vector, and demonstrate that these prices maximise revenue and welfare. The proof also implies an algorithm for finding this unique price vector.

Keywords: envy-freeness, revenue maximisation, competitive equilibrium, budget constraints, product-mix auction

1 Introduction

While a large literature, starting with Arrow and Debreu (1954), has studied markets in which buyers have no financial limitations, buyer budgets are in many cases a realistic constraint. A growing literature has been studying the implications of imposing buyer budgets on the existence and computation of equilibrium, mainly with quasilinear preferences (Conitzer et al., 2021a,b, Dobzinski et al., 2012, Murray et al., 2020). Competitive equilibrium (CE) is a standard, desirable market objective, as the fundamental theorems of welfare economics imply that a CE allocation also maximises social welfare. In many practical settings (e.g. the arctic auction (Klemperer, 2018)), however, the alternative objective of envy-free revenue maximisation, together with the option for buyers to express budget constraints, is often at least as attractive. In an envy-free allocation that maximises revenue, buyers receive bundles they demand at market prices, but the seller may prefer to allocate only a subset of her supply (Guruswami et al., 2005, Klemperer, 2008). Solutions for the two objectives of maximising revenue or welfare do not generally coincide; in many common economic settings, e.g. divisible goods markets in which buyers have concave preferences (see Section 2), the two objectives diverge significantly in the market outcome.

This paper identifies an important market setting that unifies the two objectives of envy-free revenue maximisation and welfare maximisation. Our market contains multiple buyers and one seller. The seller supplies multiple goods in finite, divisible quantities. Buyers have quasi-linear utilities and budget constraints; that is, every buyer can set an upper limit on the amount of money they wish to spend. Each buyer has a

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linear valuation, i.e. a fixed per-unit value for each good.While markets like ours are important in practice (see the discussion of ad auctions below), they have received comparatively little attention in theoretical work. Our main contribution is to demonstrate that the two objectives of social welfare and envy-free revenue coincide in quasilinear budget-constrained markets: revenue is maximised at competitive equilibrium prices. To prove this, we introduce a novel geometric object, the *feasible region*, which is defined as the set of prices at which, for every good, either the market clears or there is excess supply. This region has a smallest element which, as we show in this paper, is the revenue- and welfare-maximising price.

In applications, buyers with linear valuations are of particular interest. For example, Klemperer (2018) introduced the 'arctic auction', originally developed for the government of Iceland, in which buyers have budget-constrained preferences for multiple divisible goods. The market we consider can be interpreted as a special case of this auction. The government planned to use the arctic auction to exchange blocked accounts for other financial assets, e.g. cash or bonds of different quality, available with limited supply.[4] The auction was tailored to this use because buyers could submit a budget, as well as trade-offs between different assets through bid prices. The objective of Klemperer's auction, which is part of the family of product-mix auctions (Klemperer, 2008, 2010a, 2018), was to maximise revenue subject to envy-freeness, which is attractive for the seller (government) as well as bidders. We show in Section 3.1 that each buyer in the arctic auction has preferences corresponding to the aggregation of multiple budget-constrained quasilinear buyers with linear valuations. Although buyers in our market also have substitute preferences, the budget constraints alter the problem significantly, so that algorithms for strong-substitutes product-mix auctions (Baldwin et al., 2022, 2023) are not applicable.

Our market is also a suitable model of many ad auctions as they occur in practice. An identical setting, but with an ex-ante different notion of equilibrium, has recently been addressed by Conitzer et al. (2021a). When advertising companies compete for web space (goods) to display their ads, the decision of which publisher to choose and bid for is difficult. It may be intuitive, however, to choose an advertising budget and state demand in terms of 'limit market prices' for multiple, distinct goods, below which the seller, or the market platform, allocates those goods with the highest value for money. This is not only conceptually easier for advertisers, but also practical, feasible, and desirable from the perspective of social welfare *and* revenue, as demonstrated by our results for the budget-constrained quasilinear setting.

The market we study has recently been labelled a 'quasi-Fisher' market (Murray et al. (2020)), as it can be considered a generalisation of Fisher markets in which buyers have quasi-linear utilities. In standard Fisher markets, buyers spend their entire budget at any market prices, and so revenue is constant at all prices. In contrast, buyers with quasi-linear utility and budget constraints spend nothing when prices are unacceptably high. Hence, the notion of maximising revenue becomes a viable objective for the seller to pursue. Quasi-Fisher markets have been studied from a (robust) competitive equilibrium perspective (Chen et al., 2007, Murray et al., 2020) as well as from the perspective of Nash social welfare (Cole et al., 2017). While competitive equilibrium and Nash social welfare are desirable in terms of stability, designing an auction to maximise revenue while respecting envy-freeness may better reflect the primary interests of the seller without affecting the stability of the allocation.

Our contributions. In this paper, we show that when buyers have budget-constrained utilities with linear valuations, the two objectives of maximising revenue and finding a competitive equilibrium coincide. The unique market-clearing prices are buyer-optimal among all revenue-maximising prices in the sense that they maximise the quantities allocated to each buyer, and thus maximise buyer utilities. Building on the work of Chen et al. (2007), our results imply that efficient inner-point methods can be used to find revenue-maximising prices. Our proof proceeds geometrically and may thus convey insights of independent interest. First we show that the set of 'feasible' prices (in the sense that aggregate demand does not exceed supply) contains an elementwise-minimal price vector. This is not immediately clear, as the set of feasible prices is not convex. We then complete the argument by showing that this elementwise-minimal price vector maximises revenue *and* clears the market. In order to demonstrate the latter, we adapt a price-scaling procedure from Adsul

[4] Other related applications may include debt restructuring and the (re-)division of firms between shareholders Klemperer (2018).

et al. (2012)'s simplex-like algorithm for linear Fisher markets to our market setting. Given non-minimal feasible prices, our procedure scales down the prices of a subset of goods while maintaining feasibility.[5] Just as Adsul et al. (2012) derive a simplex-like algorithm for solving linear algorithms by repeatedly calling their price-scaling procedure, we see that applying our procedure finitely many times leads to an algorithm for finding the elementwise-minimal prices that may be more efficient and easily implementable in practice than existing interior point methods. We can then compute a corresponding envy-free allocation that assign the entire supply to buyers at these market-clearing prices using a maximum flow problem, as we describe in Section 3.2.

Organisation. The remainder of this section discusses related literature (Section 1.1) and highlights the necessity of linear valuations for our result (Section 2). In Section 3, we define our model. Our main theorem on the coincidence of revenue and welfare is stated and proved in Section 4. The price-reduction procedure is described in Section 5. In Section 6, we state our theorem on finding market-clearing prices, and Section 7 concludes.

1.1 Related literature

Computing equilibrium prices (also known as market-clearing prices) in multi-good markets has raised great interest among economists and computer scientists alike, most notably for the divisible goods market due to Léon Walras Walras (1874), for which Arrow and Debreu in their famous result Arrow and Debreu (1954) proved the existence of a general market equilibrium. Since then, much algorithmic machinery has been developed for the Arrow-Debreu market (e.g. Eaves (1976), Jain (2007)) and its important special case, the Fisher market Brainard and Scarf (2005). Fisher markets are a simple economic setting with well-known tractable algorithms for the computation of competitive market equilibria. The famous Eisenberg-Gale convex program (Eisenberg and Gale, 1959), originally introduced for linear utilities, has later been studied in the context of various types of preferences such as particular spending constraints (Birnbaum et al., 2011, Devanur and Vazirani, 2004, Garg et al., 2013) or additional transaction costs (Chakraborty et al., 2010).

Our model of budget-constrained buyers with quasilinear utilities and linear valuations can be understood as a 'quasi-Fisher' market Murray et al. (2020). In the linear Fisher setting, money has no intrinsic value: buyers do not gain any utility from any leftover budget. In contrast to the Arrow-Debreu market, prices are neither scalable in the linear nor in the quasilinear Fisher market setting. Moreover, contrasting with the linear Fisher market, in the budget-constrained buyer market with quasilinear utilities, demand is not invariant to scaling prices, in the sense that, for every buyer, there exists a limit price for each good beyond which the buyer does not demand the good.

While linear Fisher markets have received significant attention in the literature (e.g. Adsul et al. (2012), Devanur et al. (2008), Orlin (2010)), their counterpart with quasilinear utilities has appeared in various guises. The first results on quasilinear Fisher markets were developed by Chen et al. (2007). They show that competitive equilibria can be computed in polynomial time, using interior point algorithms for convex programmes. Cole et al. (2017) consider the problem of maximising Nash social welfare, mainly in linear Fisher markets spending restricted utility models, but also with extensions to Leontief, CES utility, and quasilinear utility. Bei et al. (2016) analyse Fisher markets in which buyers have budget-additive utilities, a generalisation of linear utilities in which buyers have a cap on utilities, i.e. a global satiation point. The authors devise an efficient combinatorial algorithm to compute a competitive equilibrium with a descending-price procedure. Gao and Kroer (2020) develop first-order methods for solving Fisher markets with linear, quasilinear, and Leontief utilities. They demonstrate that, for quasilinear utilities, mirror descent applied to a convex program achieves sublinear last-iterate convergence with a form of proportional response dynamics. Cheung et al. (2021) study learning dynamics in distributed economies, which can also be modelled as Fisher

[5] The procedure of Adsul et al. (2012) scales prices up instead of down. A second difference is that buyers in our setting can choose not to spend (a part of) their budget; recall that buyers in Fisher markets always spend their entire budget. This requires additional pre-processing for buyers that may be indifferent between spending all of the budget or only a fraction thereof at given prices.

markets with quasilinear utilities. For this model, the authors develop a proportional response (PR) protocol that converges to competitive equilibrium. They also develop 'chaos results' for gradient ascent dynamics in a simple setting of two firms and one good.

In contrast with this literature, our paper considers not only competitive equilibrium and social welfare as solution concept and market objective, but also maximising revenue subject to envy-freeness. Our unifying result demonstrates the importance of the quasilinear setting from a theoretical perspective as well as in applications. The practical relevance is highlighted also by a recent paper by Conitzer et al. (2021a). Their setting is particularly inspired by online ad auctions, and the basic properties of the market, buyers, goods, and budget-constrained preferences with linear values are identical to ours. However, unlike in our setting, each divisible good is sold in an independent, single-unit first-price auction, in which only the highest bidders can win a positive quantity. Conitzer et al. (2021a) introduce the solution concept of *first price pacing equilibria* (FPPE), in which the bids submitted correspond to the buyers' values scaled (uniformly for each buyer) by a multiplicative factor, the pacing multiplier.[6] Interestingly, this at first sight unrelated auction procedure can also be solved using the modified Eisenberg-Gale convex programme of Chen et al. (2007). Moreover, Conitzer et al. (2021a) show that the unique FPPE corresponds to a competitive equilibrium in the sense of our setting; that is, in the overarching market for all goods with budget-constrained, quasilinear buyers. While the authors show that the FPPE is revenue-maximal among all budget-feasible pacing multipliers and corresponding allocations, our work implies that the FPPE is indeed revenue-maximising subject to envy-freeness in the entire market.

2 Motivating examples: when revenue and welfare fail to coincide

The purpose of the first example is to demonstrate the difficulties one encounters with diminishing marginal values and motivates our interest in linear valuations. We also introduce some of the geometric properties of our framework in a two-good example with linear valuation, which are natural to work with for our objective.

Example 1. A seller (she) can provide a single good available with supply $s \in \mathbb{R}_+$ at zero marginal costs. There is one buyer (he) with a continuous valuation $v : \mathbb{R} \to \mathbb{R}$ for the good. The buyer has quasi-linear utility $u(p, x) = v(x) - px$, where p denotes the unit price of the good, and a budget $\beta \in \mathbb{R}_+ \cup \{\infty\}$. We aim to find an allocation $x \leq s$ and a price p that maximise social welfare (SW) or the seller's revenue (R). The buyer demands quantity x at price p if $x \in \arg\max_x v(x) - px$ such that $px \leq \beta$.

The revenue maximisation problem is constrained in the sense that the seller cannot enforce a specific allocation to the buyer. She can only set a price, anticipating the buyer's demand. Thus, the seller's revenue is maximised at $(x, p) \in \arg\max_{x,p} px$ such that $x \in \arg\max_x v(x) - px$ and $x \leq s$. Social welfare, on the other hand, is simply maximised at $(x, p) \in \arg\max_{x,p} v(x) - px$ such that $x \leq s$.

To derive optimal (for revenue or social welfare) allocations and prices, we first need to make some assumptions on the buyer's valuation. In the economics literature, diminishing marginal values are a typical assumption on preferences. In that case, the buyer always demands a quantity such that his marginal utility at this quantity is zero. Social welfare maximisation then implies $v'(x) = p$ and $x = s$; therefore, social-welfare maximising prices are always 'market-clearing'.[7] However, if the seller were allowed to adjust the price subject to respecting the buyer's demand, she might be able to not sell the entire supply and extract more revenue. The following proposition shows that this is indeed the case for all strongly concave value functions if the buyer's budget is large.

Proposition 1. *Let v be differentiable and strongly concave with parameter m for some $m > 0$, and let $\beta = \infty$. Then there exists some supply $s \in \mathbb{R}$ so that revenue is not maximised at the market-clearing price.*

[6] An FPPE is defined as a set of pacing multipliers (one for each buyer) and allocations that satisfy the allocation and pricing rule of standard first-price auctions, as well as budget feasibility, supply feasibility, market clearing for demanded goods, and 'no unnecessary pacing', i.e. a buyer's multiplier equals one if she has unspent budget.

[7] The welfare theorems hold even in the general version of our model with multiple goods, as we show in Section 3.

Proof. Recall that at any price p, the buyer demands the bundle x that maximises $v(x)-p$. Given valuation $v(x)$, revenue is maximised at $(x,p) \in \arg\max_{x,p} px$ such that $v'(x)=p$ and $x \leq s$. Thus, maximal revenue given v and s is $v'(x)x$ for some $x \leq s$. Social welfare is maximised at $(x,p) \in \arg\max_{x,p} v(x)-px$ such that $x \leq s$, i.e. revenue at the social optimum is $v'(s)s$. We show that there exists x and s with $x < s$ and $v'(x)x > v'(s)s$. Recall that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is strongly concave if it satisfies $|f'(x) - f'(y)| \geq m\|x-y\|$ for all distinct $x, y \in \mathbb{R}$ (v does not need to be twice differentiable).

First, note that there exists $\overline{x} < \infty$ such that $v'(x) < mx$ for all $x \geq \overline{x}$, due to strict concavity of v. Fix some supply $s \geq \overline{x}$ and let $\varepsilon = \frac{1}{2}(ms - v'(s))$. As v is strongly concave, we also have $v'(s-\varepsilon) \geq v'(s) + m\varepsilon$ for any ε. Hence, $v'(s-\epsilon)(s-\epsilon) \geq (v'(s)+m\epsilon)(s-\epsilon) = v'(s)s + \epsilon(ms - v'(s) - m\epsilon) > v'(s)s$. ◻

When the buyer has a finite budget, the situation is more intricate. Indeed, supply can only lie in the finite interval $X := [0, \max\{x \mid xv'(x) \leq \beta\}]$. We define $\tilde{x}$ implicitly by $\tilde{x}v'(\tilde{x}) = \beta$ and we call any bundle $x \in X$ *budget-feasible.*

Proposition 2. *Suppose the buyer has a strongly concave valuation v with parameter m and a finite budget β. If supply s is contained in X with $v'(s) < ms$, or if $m > \frac{v'(\tilde{x})}{\tilde{x}}$, then revenue is not maximised at the market-clearing price.*

Proof. If there exists supply $s \in X$ with $v'(s) < ms$, then the result follows analogously to the proof of Proposition 1. We now prove the second part of the statement. The budget constraint is given by $px \leq \beta$. At the demanded quantity, $v'(x) = p$ holds. Thus, for all feasible x, it must hold that $v'(x) \leq \beta/x$, so $x \leq \tilde{x}$. Now we demonstrate that $v'(\tilde{x}-\epsilon) \leq m(\tilde{x}-\epsilon)$ for some small ϵ. Then the result follows from Proposition 2. For some $\delta > 0$, we have

$$m(\tilde{x}-\epsilon) \geq \left(\frac{v'(\tilde{x})}{\tilde{x}} + \delta\right)(\tilde{x}-\epsilon) = \frac{(v'(\tilde{x}) + \delta\tilde{x})(\tilde{x}-\epsilon)}{\tilde{x}} \geq v'(\tilde{x}-\epsilon).$$

The last inequality holds for $\epsilon \to 0$ and some $\delta > 0$. ◻

Proposition 2 tells us that for any strongly concave valuation, we can find a combination of budget and supply such that the maximisers of the social welfare and the revenue maximisation problem do not coincide. For example, we may require the valuation to be sufficiently concave relative to the budget. Example 2 shows a specific valuation for a single good for which revenue and welfare do not coincide. Note that the budget constraint never binds, so this example also applies to general quasi-linear utilities without budget constraints.

Example 2. Consider an auction with a single good available in $s = 3$ units that has a single buyer. The buyer has valuation $v : \mathbb{R} \to \mathbb{R}$ given by $v(x) = \frac{4}{\log 2}(1-2^{-x})$ and budget 2. Then revenue is not maximised at market-clearing prices. Indeed, note that the utility of the buyer for quantity q at price p is $u(x,p) = v(x) - px$, so the buyer's demand $D(p)$ at p is found by solving $v'(x) = p$, which yields $x = -\log_2(\frac{p}{4})$. At $p = 0.5$, we have demand $x = 3$, so p clears the market. Revenue at p is $px = 1.5$. At price $q = 1$, we have demand $y = 2$, so q does not clear the market, but revenue is $qy = 2$, which is greater. Revenue is maximised at $p = \frac{4}{e}$ with a demanded quantity of $\frac{1}{\log(2)}$ and a revenue of $\frac{4}{e\log(2)}$.

In light of the above examples and propositions, we consider the class of constant marginal values, i.e. $v(x) = vx$.

Proposition 3. *Let the buyer's valuation be $v(x) = vx$ with budget β. Then the seller's revenue is maximised at market-clearing prices.*

The above proposition is straightforward. Social welfare is maximised at $p = \min\{v, \frac{\beta}{s}\}$ and $x = s$. Because the buyer spends his entire budget, the seller cannot extract more revenue, and the welfare maximising allocation and price are also revenue-optimal.

An immediate question to ask is whether this simple logic extends to more general environments and preferences. The answer we present in this paper is affirmative: if any number of buyers have quasi-linear, budget-constrained utility and linear values for any number of goods, then revenue and social welfare are maximised at a unique set of elementwise-minimal prices. This result, however, is not immediate. In the following, we illustrate the difficulty in another simple example with two goods.

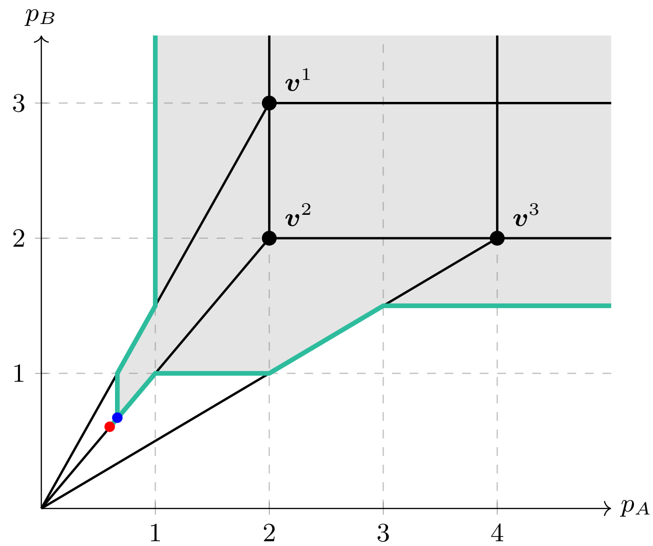

Fig. 1: The feasible region in price space corresponding to Example 3.

Example 3. Two goods A and B are for sale with a supply of $s_A = 3$ and $s_B = 2$. There are three buyers $1, 2$, and 3 with the following marginal values: $\boldsymbol{v}^1 = (v_A^1, v_B^1) = (2,3)$, $\boldsymbol{v}^2 = (v_A^2, v_B^2) = (2,2)$, and $\boldsymbol{v}^3 = (v_A^3, v_B^3) = (4,2)$. Utilities are quasi-linear, i.e. $u^i(x,p) = \sum_j (v_j^i - p_j)x_j$ for buyer i. Each buyer has a budget of $\beta^1 = \beta^2 = \beta^3 = 1$. We can allocate the 3 units of A and 2 units of B among the three buyers to maximise either revenue or social welfare, but need to respect individual demand. It is not hard to check that at given prices (p_A, p_B) each buyer i will demand a good $j \in \arg\max_{j=1,2} \frac{v_j^i}{p_j}$ if $v_j^i \geq p_j$. Any good k with $v_k^i < p_k$ will never be demanded by buyer i. This kind of individual demand can be easily represented in price space (more detail in Section 3.1). At some prices, aggregate demand is too large to be satisfied by supply. Prices at which aggregate demand does *not* exceed supply are called *supply-feasible.* The set of supply-feasible prices makes up the *feasible region.* The bids (black dots) and the feasible region (in grey) are illustrated in Fig. 1. Note that the feasible region also includes a short line segment between $\boldsymbol{p}^* := (\frac{3}{5}, \frac{3}{5})$ (red dot) and $\boldsymbol{p}' := (\frac{2}{3}, \frac{2}{3})$ (blue dot). The feasible region has the key property that for any pair of feasible price vectors, the elementwise minimum of them also belongs to the region.

At prices $(\frac{3}{5}, \frac{3}{5})$, buyer 1 demands $\frac{5}{3}$ of B, buyer 3 demands $\frac{5}{3}$ of A, and buyer 2 demands $x_A^2 \in [0, \frac{5}{3}]$ copies of A and $\frac{5}{3} - x_A^2$ of B. With supply $(s_A, s_B) = (3, 2)$, set $x_A^2 = \frac{4}{3}$ to clear the market. It is easy to check that indeed any prices on $[p^*, p']$ induce a feasible allocation. All prices $[p^*, p']$ are revenue-maximising. However, only $\boldsymbol{p}^*$ clears the market and constitutes CE prices.

3 The market, preferences, and objectives

Preliminaries. For any two n-dimensional vectors $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{w} \in \mathbb{R}^n$, we write $\boldsymbol{v} \leq \boldsymbol{w}$ when the inequality holds element-wise. For any $j \in [n]$, $\boldsymbol{e}^j$ denotes the vector whose j-th entry is 1 and other entries are 0. For convenience, we also define $\boldsymbol{e}^0 = \boldsymbol{0}$. The dot product of $\boldsymbol{v}$ and $\boldsymbol{w}$ is denoted $\boldsymbol{v} \cdot \boldsymbol{w}$. For any function $f : A \times B \to \mathbb{R}^n$, we use the implicit summation $f(A', b) = \sum_{a \in A'} f(a, b)$, $f(a, B') = \sum_{b \in B'} f(a, b)$, and $f(A', B') = \sum_{a \in A', b \in B'} f(a, b)$, for any $A' \subseteq A$ and $B' \subseteq B$.

Our Market. There are n goods $[n] := \{1, \ldots, n\}$, typically denoted by j and k. We will also work with a notional *null good* 0, and let $[n]_0 = \{0, \ldots, n\}$. A *bundle* of goods, typically denoted by $\boldsymbol{x}$ or $\boldsymbol{y}$ in this paper, is a vector in $\mathbb{R}^n$ whose j-th entry denotes the *quantity* of good j. The seller has a supply bundle $\boldsymbol{s} \in \mathbb{R}^n$ that they wish to sell, partially or completely, by setting uniform market prices $\boldsymbol{p} \in \mathbb{R}^n_+$; a price vector $\boldsymbol{p} \in \mathbb{R}^n$

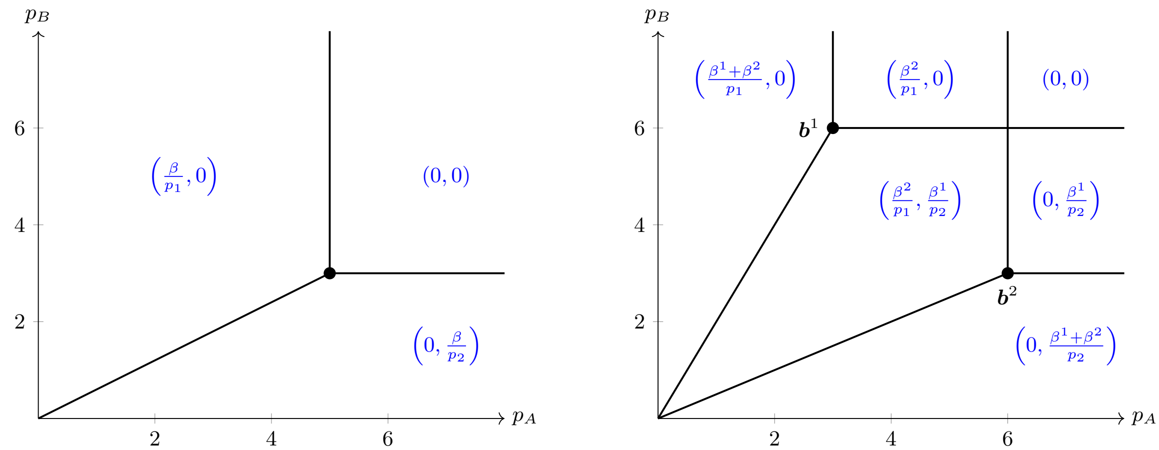

Fig. 2: Illustrations of (aggregate) demand correspondences for buyers with quasi-linear utilities, linear valuations and budget constraints. Left: Demand for a single buyer with linear valuation $\boldsymbol{v} = (5, 3)$ and budget constraint β divides price space into three regions. At prices that are element-wise larger than $\boldsymbol{v}$, the buyer demands nothing. At prices $\boldsymbol{p}$ that lie above the line connecting point $\boldsymbol{v}$ to the origin, the buyer demands a quantity β/p_1 of good 1. Conversely, at prices below the line, the buyer demands β/p_2 of good 2. Right: The aggregate demand correspondence of two buyers: the first has valuation $\boldsymbol{v}^1 = (3, 6)$ and budget β^1, and the second has valuation $\boldsymbol{v}^2 = (6, 3)$ and budget β^2.

has a price entry for each of the n goods. When working with the notional null good 0, we implicitly define $p_0 = 0$.

We have m buyers $[m] = \{1, \ldots, m\}$, typically denoted by i, with budgets β^i who compete for the supply of the n distinct goods. Throughout this paper, we assume that every buyer $i \in [m]$ has a *linear valuation* $v^i : \mathbb{R}^n \to \mathbb{R}$ and budget β^i. That is, each buyer i possesses a vector $\boldsymbol{v}^i \in \mathbb{R}^n_+$ so that the valuation is given by $v^i(\boldsymbol{x}) = \boldsymbol{v}^i \cdot \boldsymbol{x}$ for all bundles $\boldsymbol{x} \in \mathbb{R}^n$. Moreover, without loss of generality we can assume that, for every good $j \in [n]$, there exists at least one buyer with a positive valuation v_j for the good. (Otherwise we can simply remove the good from the auction.) Utilities are quasi-linear, i.e. the buyer derives utility $u^i(\boldsymbol{x}, \boldsymbol{p}) = v^i(\boldsymbol{x}) - \boldsymbol{p} \cdot \boldsymbol{x} = (\boldsymbol{v}^i - \boldsymbol{p}) \cdot \boldsymbol{x}$ from receiving bundle $\boldsymbol{x} \in \mathbb{R}^n$ at prices $\boldsymbol{p} \in \mathbb{R}^n$. At any given prices $\boldsymbol{p}$, a buyer i demands the bundles $\boldsymbol{x}$ that maximise their utility $u^i(\cdot, \boldsymbol{p})$, subject to not exceeding their budget (expressed by the budget constraint $\boldsymbol{x} \cdot \boldsymbol{p} \leq \beta^i$). This leads to the *demand correspondence* $D^i(\boldsymbol{p}) = \max_{\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \cdot \boldsymbol{p} \leq \beta^i} (v^i(\boldsymbol{x}) - \boldsymbol{p} \cdot \boldsymbol{x}) = \max_{\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \cdot \boldsymbol{p} \leq \beta^i} (\boldsymbol{v}^i - \boldsymbol{p}) \cdot \boldsymbol{x}$.

As with prices, we define $v_0^i = 1$ for the value of the null good. Moreover, for any buyer i and prices $\boldsymbol{p}$, we define the *demanded good set* $J^i(\boldsymbol{p}) = \arg\max_{j \in [n]_0} \frac{v_j^i}{p_j}$. This set of goods is also referred to in the literature on Fisher markets as maximising *bang-per-buck*. (Note that $J^i(\boldsymbol{p})$ contains the null good if $\max_{j \in [n]_0} \frac{v_j^i}{p_j} = 1$, as we have $v_0^i = 1$ and $p_0 = 1$ by definition.) This allows us to provide an alternative characterisation of the demand correspondence that we rely on when proving our results below. In particular, Lemma 1 makes the initial observation that bundles only contain quantities of goods that maximise bang-per-buck; in other words, if $\boldsymbol{x}$ is a bundle demanded by buyer i at $\boldsymbol{p}$, then $x_j > 0$ implies $j \in J^i(\boldsymbol{p})$. Moreover, any demanded bundle is the convex combination of the 'extremal' bundles that arise when the entire budget is spent on a single demanded good in $J^i(\boldsymbol{p})$. In Section 3.1, we will see that the arctic product-mix auction introduces a bidding language which starts from this definition of demand to characterise a more general demand type.

Lemma 1. *Fix $\boldsymbol{p} \in \mathbb{R}^n_+$. Let $\boldsymbol{x} \in D(\boldsymbol{p})$ for some buyer with linear valuation $\boldsymbol{v}$, budget β and demanded good set $J(\boldsymbol{p})$. If $x_j > 0$, then $j \in J(\boldsymbol{p})$. Moreover $D(\boldsymbol{p}) = \operatorname{conv}\left\{\frac{\beta}{p_j}\boldsymbol{e}^j \mid j \in J(\boldsymbol{p})\right\}$.*

Proof. Suppose, for contradiction, we have a non-null good j with $x_j > 0$ and $j \notin J(\boldsymbol{p})$. Then there exists another good $k \in J(\boldsymbol{p})$ with $\frac{v_j}{p_j} < \frac{v_k}{p_k}$. If k is the null good, then we define a new bundle $\boldsymbol{x}'$ from $\boldsymbol{x}$ by not spending on good j, so $\boldsymbol{x}' = \boldsymbol{x} - x_j \boldsymbol{e}^j$. It is immediate that $\boldsymbol{x}'$ is budget-feasible in the sense that it does not exceed the buyer's budget β at $\boldsymbol{p}$, as $\boldsymbol{x}' \cdot \boldsymbol{p} < \boldsymbol{x} \cdot \boldsymbol{p} \leq \beta$. Moreover, its utility is $(\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x}' = (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x} + x_j(p_j - v_j) > (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x}$. The last inequality follows from our assumption that $\frac{v_j}{p_j} < \frac{v_0}{p_0} = 1$, and it contradicts our assumption that $\boldsymbol{x}$ maximises utility. Now suppose k is not the null good. Then we let $\boldsymbol{x}'$ be the bundle obtained from $\boldsymbol{x}$ by spending nothing on good j and instead using the same amount of money to buy (additional) quantities of good k, so $\boldsymbol{x}' \coloneqq \boldsymbol{x} + x_j \left(\frac{p_j}{p_k} \boldsymbol{e}^k - \boldsymbol{e}^j \right) \geq 0$. Then $\boldsymbol{x}'$ does not exceed the buyer's budget, as we can verify $\boldsymbol{p} \cdot \boldsymbol{x}' = \boldsymbol{p} \cdot \boldsymbol{x} \leq \beta$. Moreover, its utility is $(\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x}' = (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x} + x_j \left(\frac{p_j}{p_k} v_k - v_j \right) = (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x} + x_j p_j \left(\frac{v_k}{p_k} - \frac{v_j}{p_j} \right) > (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x}$.

Next we consider the second claim $D(\boldsymbol{p}) = \operatorname{conv} \left\{ \frac{\beta}{p_j} \boldsymbol{e}^j \mid j \in J(\boldsymbol{p}) \right\}$. Fix $\boldsymbol{x} \in D(\boldsymbol{p})$, and let β_j denote, for each good $j \in [n]_0$, the amount that the buyer spends on j to obtain x_j. If $0 \in J(\boldsymbol{p})$, then we let $\beta_0 = \beta - \beta_{[n]}$ be the remainder of the budget that the buyer doesn't spend. Hence we have $\beta_{J(\boldsymbol{p})} = \beta$ and $\beta_j = 0$ for $j \notin J(\boldsymbol{p})$. Moreover, $x_j = \frac{\beta_j}{p_j}$ for all $j \in [n]$, and thus $\boldsymbol{x} = \sum_{j \in J(\boldsymbol{p})} \frac{\beta_j}{\beta} \frac{\beta}{p_j} \boldsymbol{e}^j$, so $\boldsymbol{x}$ lies in the convex hull. Now we show that all bundles in the convex hull lie in $D(\boldsymbol{p})$. Note that reallocating spending β_j on good j to another good $k \in J(\boldsymbol{p})$ leads to another bundle, the bundle $\boldsymbol{x}'$ as defined above, that does not exceed the buyer's budget and maximises utility. The latter follows as $\frac{v_k}{p_k} = \frac{v_j}{p_j}$ for $j, k \in J(\boldsymbol{p})$. Repeatedly reallocating spending until the bundle only contains positive quantities of a single good $j \in J(\boldsymbol{p})$ shows that the vertices of the convex hull are in $D(\boldsymbol{p})$. Finally, note that all bundles in the convex hull are convex combinations of the vertices, and taking convex combinations retains both budget-feasibility and maximal utility. ☐

Expenditure and allocation. When solving the auction, the seller sets market prices and determines an allocation to each buyer. Note that the allocation and prices directly control how much each buyer spends on each good. We capture this expenditure formally, as follows. An expenditure function $e : [m] \times [n]_0 \to \mathbb{R}_+$ expresses, for every buyer $i \in [m]$ and every good $j \in [n]_0$, the amount $e(i,j)$ that buyer i spends on good j. For each expenditure, we can define the corresponding *allocation* $\pi : [m] \times [n]_0 \to \mathbb{R}_+$ denoting the quantity $\pi(i,j) = \frac{e(i,j)}{p_j}$ of good j that buyer i receives. Moreover, for notational convenience we can define vectors $e(i)$ and $\pi(i)$ for each buyer i as $e(i)_j = e(i,j)$ and $\pi(i)_j = \pi(i,j)$ for all goods $j \in [n]$. The vector $e(i)$ captures how much buyer i spends on each good, while $\pi(i)$ denotes the bundle of goods that i receives. In particular, implicit summation allows us to express aggregate spending and demand as $e([m])$ and $\pi([m])$.

Given an expenditure e, we say that good $j \in [n]$ is *depleted* if $e([m], j) = p_j s_j$, and *allocable* if $e([m], j) < p_j s_j$. If the seller wishes not to exceed supply and to allocate each buyer a bundle they demand at the chosen market prices, she needs to satisfy three criteria that are expressed in Definition 1.

Definition 1 (Feasibility and validity). *Let $e : [m] \times [n]_0 \to \mathbb{R}_+$ be an expenditure. Then e is:*

(i) Supply-feasible *at prices $\boldsymbol{p}$ if supply weakly exceeds the aggregate allocation for all goods j, so $e([m], j)/p_j \leq s_j$ for all $j \in [n]$.*
(ii) Budget-feasible *at prices $\boldsymbol{p}$ if every buyer i spends weakly less than its budget on non-null goods. The remaining (non-negative) amount is always spent on the null good, so $e(i, [n]_0) = \beta^i$ for all buyers $i \in [m]$.*
(iii) Demand-valid *at prices $\boldsymbol{p}$ if each buyer only spends on goods that maximise her bang-per-buck, so $e(i,j) > 0$ implies $j \in J^i(\boldsymbol{p})$.*

An expenditure is feasible at $\boldsymbol{p}$ if (i) and (ii) hold, and it is valid at $\boldsymbol{p}$ if (iii) also holds. We extend the same terminology to the allocations π corresponding to feasible or valid expenditures defined by $\pi(i,j) = \frac{e(i,j)}{p_j}$. Note that for any valid allocation π at prices $\boldsymbol{p}$, the bundle of goods allocated to buyer i under π lies in $D^i(\boldsymbol{p})$.

Definition 2. *Prices $\boldsymbol{p}$ are feasible if there exists an expenditure e that is valid at $\boldsymbol{p}$.*

In Section 4.1, we show that the set of feasible prices forms a lower semi-lattice, and in particular has an *elementwise-minimal price vector* that is dominated by all other feasible prices.

Definition 3. *A feasible allocation π is efficient if, for any feasible allocation $\pi' \neq \pi$, we have*

$$\sum_{i\in[m],j\in[n]} v_j\pi(i,j) \geq \sum_{i\in[m],j\in[n]} v_j\pi'(i,j).$$

Proposition 4. *Suppose π is a budget-feasible allocation at prices $\boldsymbol{p}$, and fix a buyer i. Then we have $\sum_{j\in[n]_0}(v^i_j - p_j)\pi(i,j) \geq \sum_{j\in[n]_0}(v^i_j - p_j)\pi'(i,j)$ for all budget-feasible allocations π' if, and only if, $\pi(i) \in D^i(\boldsymbol{p})$.*

Proof. This follows immediately from budget-constrained quasi-linearity. ◻

Market objectives. We consider three separate objectives for the seller. She may set prices with the objective of maximising her revenue, or she may set prices to maximise social welfare. Intuitively (and we prove this below), at the social optimum, the seller's entire supply is allocated. To maximise revenue, the seller may prefer to raise prices and retain some of her supply.

1. **Envy-free revenue maximisation.** Given a valid allocation π at prices $\boldsymbol{p}$, the revenue for the seller is given by $\boldsymbol{p}\cdot\pi([m]) = e([m])$. Hence, we define the indirect revenue function as $R(\boldsymbol{p}) := \max_\pi \boldsymbol{p}\cdot\pi([m])$ subject to π being valid. A solution to the *envy-free revenue maximisation problem* consists of a price vector $\boldsymbol{p}$ and allocation π that maximises $R(\boldsymbol{p})$.[8]
2. **Welfare maximisation.** Recall that the buyers are price takers. Thus, social welfare is defined with respect to the submitted values. A solution to the *social welfare maximisation problem* is a price vector and allocation $(\boldsymbol{p},\pi)$ that maximises social welfare $\sum_{i\in[m]}\boldsymbol{v}^i\cdot\pi(i)$ subject to π being a valid allocation at $\boldsymbol{p}$.
3. **Competitive equilibrium.** Given a list of buyers $[m]$ and a supply vector $\boldsymbol{s}$, prices $\boldsymbol{p}$ and allocation π (at $\boldsymbol{p}$) form a competitive equilibrium if, and only if, π is valid at $\boldsymbol{p}$ and $\pi([m],j) = s_j$ for all goods j with $p_j > 0$. In this case, we refer to $\boldsymbol{p}$ as *competitive equilibrium prices* or *market-clearing prices*.

We show that the first and second welfare theorem hold in our market: competitive equilibrium maximises welfare, and any efficient allocation can be supported by prices to constitute a competitive equilibrium. Therefore, we focus on the two objectives of envy-free revenue maximisation and competitive equilibrium.

Theorem 1. *If $(\boldsymbol{p},\pi)$ is a competitive equilibrium, π is an efficient allocation.*

Proof. Let $(\boldsymbol{p},\pi)$ be a competitive equilibrium with corresponding expenditure e. Let π' be a feasible allocation at $\boldsymbol{p}$ with expenditure e'. Proposition 4 implies

$$\sum_{i\in[m],j\in[n]} (v^i_j - p_j)\pi(i,j) \geq \sum_{i\in[m],j\in[n]} (v^i_j - p_j)\pi'(i,j).$$

Rearranging and using the fact that $e([m],[n]) = \boldsymbol{p}\cdot\boldsymbol{s} \geq e'([m],[n])$, we have

$$\sum_{i\in[m],j\in[n]} v^i_j\pi(i,j) - \sum_{i\in[m],j\in[n]} v^i_j\pi'(i,j) \geq e([m],[n]) - e'([m],[n]) \geq 0.$$

◻

Theorem 2. *Let $(\boldsymbol{p},\pi)$ be a competitive equilibrium and π' be another efficient and feasible allocation at some prices. Then $(\boldsymbol{p},\pi')$ is also a competitive equilibrium.*

[8] In the remainder of the paper, we occasionally omit 'envy-free' in the term 'envy-free revenue maximisation' for brevity. However, revenue maximisation in this work is subject to the envy-freeness constraint that allocations are budget-feasible and demand-valid throughout.

Proof. By definition of efficiency, we first note that

$$\sum_{i\in[m],j\in[n]} v^i_j \pi(i,j) = \sum_{i\in[m],j\in[n]} v^i_j \pi'(i,j). \tag{1}$$

Secondly, as π is a competitive equilibrium and π' is feasible, we have $\pi([m]) = \boldsymbol{s}$ and $\pi'([m]) \leq \boldsymbol{s}$. Multiplying both terms with $\boldsymbol{p}$, we get

$$\boldsymbol{p} \cdot \pi([m]) \geq \boldsymbol{p} \cdot \pi'([m]). \tag{2}$$

We now show that equality holds for (2). Indeed, if we suppose that (2) holds with strict inequality and use Proposition 4, we obtain

$$\begin{aligned}\sum_{i\in[m],j\in[n]} v^i_j \pi(i,j) &= \sum_{i\in[m],j\in[n]} (v^i_j - p_j)\pi(i,j) + \boldsymbol{p}\cdot\pi([m]) \\ &> \sum_{i\in[m],j\in[n]} (v^i_j - p_j)\pi'(i,j) + \boldsymbol{p}\cdot\pi'([m]) \\ &= \sum_{i\in[m],j\in[n]} v^i_j \pi'(i,j).\end{aligned}$$

□

3.1 Arctic auctions

The market we consider can be interpreted as an important special case of the arctic product-mix auction market first proposed by Paul Klemperer (Klemperer, 2018) for the government of Iceland.[9] In this market, the seller has a given supply $\boldsymbol{s} \in \mathbb{R}^n_+$ of multiple divisible goods and wishes to find a valid allocation of some subset of this supply among a finite set of buyers with the goal of maximising revenue. The buyers express their demand preferences by submitting a collection of 'arctic bids', each of which consisting of an n-dimensional vector $\boldsymbol{b} \in \mathbb{R}^n_+$ and a monetary budget $\beta(\boldsymbol{b})$. At a market price *above the stated bid price*, an arctic bid *rejects the good*. At market prices below the stated bid prices, the bid spends its budget on goods that yield the highest 'bang-per-buck', i.e. the highest ratio of value to price $\frac{b_j}{p_j}$. When multiple goods maximise 'bang-per-buck', the bid can arbitrarily divide its expenditure between these goods. Moreover, if the maximal bang-per-buck is 1, then the bid may choose not to spend part of its budget (which we interpret as spending on the null good). Hence, Lemma 1 implies that each arctic bid $\boldsymbol{b}$, interpreted in isolation, induces a quasi-linear demand with linear valuation $v(\boldsymbol{x}) = \boldsymbol{b}\cdot\boldsymbol{x}$ and budget $\beta(\boldsymbol{b})$ as defined in Section 3. Moreover, if we denote the demand correspondence of bid $\boldsymbol{b}$ by $D_{\boldsymbol{b}}$, then the demand correspondence of a collection $[m]$ of bids is defined by the Minkowski sum of demands $D_{\mathcal{B}}(\boldsymbol{p}) = \left\{\sum_{\boldsymbol{b}\in[m]} \boldsymbol{x}^{\boldsymbol{b}} \mid \boldsymbol{x}^{\boldsymbol{b}} \in D_{\boldsymbol{b}}(\boldsymbol{p})\right\}$. Equivalently, $D_{\mathcal{B}}$ can be understood as the aggregate demand of $|\mathcal{B}|$ buyers with quasi-linear demand, linear valuations $\boldsymbol{b}\in\mathcal{B}$ and budgets $\beta(\boldsymbol{b})$. By submitting multiple arctic bids, buyers can express richer preferences. Nevertheless, it is straightforward to see from the definition of $D_{\mathcal{B}}$ that for the purposes of solving the auction (for welfare or for revenue), the seller can treat each bid independently, and we can assume without loss of generality that each buyer submits a single bid. Hence our main result on the coincidence of welfare and envy-free revenue holds also for the special case of the arctic auction we describe.

3.2 Computing allocations

We briefly describe how to compute a valid allocation that clears supply, given market-clearing prices $\boldsymbol{p}^*$. This allocation forms a competitive equilibrium together with $\boldsymbol{p}^*$, and also maximises revenue among all

[9] The arctic product-mix auction is a variant of the original product-mix auction developed for the Bank of England by Paul Klemperer Klemperer (2008, 2010b, 2018). In the general arctic product-mix auction, the seller can additionally choose cost functions to fine-tune its preferences, while in our model, we assume the seller's costs to be zero. See (Fichtl, 2022) for some discussion of the general case.

allocations at $\boldsymbol{p}^*$. In order to find the allocation, which exhausts supply in all goods, consider a maximum flow problem on a network with nodes consisting of a source s, sink t, the goods $[n]_0$ (the goods $[n]$ together with the notional null good 0), and the buyers $[m]$. For each buyer $i \in [m]$, we have an arc from the source to i with a flow capacity equal to the buyer's budget β^i. For each buyer-good pair (i, j), with $i \in [m]$ and $j \in [n]_0$, we have an arc (with unbounded capacity) from i to j if the buyer i demands good j at prices $\boldsymbol{p}$. Finally, we have a arc from each good $j \in [n]_0$ to the sink, with flow capacity $s_j * p^*_j$. For the null good 0, the capacity is unbounded. These capacities corresponds to the maximal amount that can be spent on the goods at prices $\boldsymbol{p}^*$ without exceeding supply.

It is straightforward to see that feasible flows in the network correspond to valid expenditures in our market. As $\boldsymbol{p}^*$ is market-clearing, there exists an expenditure that exhausts supply, and this corresponds immediately to a flow that saturates all the edges to the target node. Hence any maximum flow leads to an allocation that exhausts supply in all goods, as required. Moreover, this allocation can be found in polynomial time (under the assumption that $\boldsymbol{p}^*$ is known) with one of the many polynomial-time algorithms for maximum flow problems.

4 The revenue-welfare coincidence

Theorem 3. *If buyers have quasi-linear utility and linear valuations, revenue and welfare are maximised at the unique elementwise-minimal feasible price vector $\boldsymbol{p}^*$.*

To prove this in our general setting, we proceed as follows:

1. A key property of the feasible region is that it forms a lower semi-lattice: that is, there exists an elementwise-minimal price vector that is dominated by all other feasible prices. We show this in Section 4.1 by first proving that, for any two feasible prices, the element-wise minimum of these prices also belongs to the region.
2. In Section 4.2, we prove that revenue is maximised at elementwise-minimal feasible prices. Note that revenue may also be maximised at other prices.
3. Section 5 provides a procedure that, given prices and a valid allocation with allocable goods, reduces prices and finds a new valid allocation. Our procedure is an adaptation of Adsul et al. (2012)'s price-scaling procedure designed for Fisher markets. In particular, this price-scaling procedure implies that the elementwise-minimal feasible prices $\boldsymbol{p}^*$ clear the market. Hence both social welfare and revenue (subject to envy-freeness) are maximised at $\boldsymbol{p}^*$.
4. Finally, Proposition 5 shows that market-clearing prices are unique.

4.1 Elementwise-minimal prices

Recall that prices $\boldsymbol{p}$ are feasible if there exists a valid allocation π at $\boldsymbol{p}$. We show that set of feasible prices form a lower semi-lattice. In particular, there exists a special price vector $\boldsymbol{p}^*$ that is element-wise smaller than all other feasible prices (so that $\boldsymbol{p}^* \leq \boldsymbol{p}$ for all feasible $\boldsymbol{p}$). For any two price vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$, we let $\boldsymbol{p} \wedge \boldsymbol{p}'$ denote their *element-wise minimum* defined as $(\boldsymbol{p} \wedge \boldsymbol{p}')_i = \min\{p_i, p'_i\}$. The following lemma is central to our proof.

Lemma 2. *If $\boldsymbol{p}$ and $\boldsymbol{p}'$ are feasible, then so is their element-wise minimum $\boldsymbol{p} \wedge \boldsymbol{p}'$.*

Fix feasible prices $\boldsymbol{p}$ and $\boldsymbol{p}'$ with element-wise minimum $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{p}'$, and let π and π' respectively denote valid allocations at $\boldsymbol{p}$ and $\boldsymbol{p}'$. In order to prove Lemma 2, we construct an allocation τ at $\boldsymbol{r}$ and show that it is valid. We first define the set of goods A in which $\boldsymbol{p}$ is strictly dominated by $\boldsymbol{p}'$, and its complement B, so $A = \{j \in [n] \mid p_j < p'_j\}$ and $B = \{j \in [n] \mid p_j \geq p'_j\}$. Then our allocation τ at $\boldsymbol{r}$ is given by

$$\tau(i, \cdot) = \begin{cases} \pi'(i, \cdot) & \text{if buyer } i \text{ demands some good } j \in B \text{ at } \boldsymbol{r}, \\ \pi(i, \cdot) & \text{otherwise.} \end{cases} \tag{3}$$

In order to prove the validity of τ, we first state a technical lemma that establishes the connection between a buyer's demand at $\boldsymbol{p}$, $\boldsymbol{p}'$ and $\boldsymbol{r}$.

Lemma 3. *Suppose buyer i demands good $j \in A$ at $\boldsymbol{r}$. Then they also demands j at $\boldsymbol{p}$ and $J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r})$. Similarly, suppose buyer i demands $j \in B$ at $\boldsymbol{r}$. Then they also demand j at $\boldsymbol{p}'$, and $J^i(\boldsymbol{p}') \subseteq J^i(\boldsymbol{r})$. Moreover, we have $J^i(\boldsymbol{p}') \subseteq B$.*

Proof. Fix a buyer i who demands good $j \in A$ at $\boldsymbol{r}$. As $p_j = r_j$, this implies $\frac{v_j^i}{p_j} = \frac{v_j^i}{r_j} \geq \frac{v_j^i}{r_j} \geq \frac{v_j^i}{p_j}$ for all goods $j \in [n]_0$. The first inequality holds due to the definition of demand, and the second inequality follows from $r_j \leq p_j, \forall j \in [n]_0$. Hence, the buyer demands good j at $\boldsymbol{p}$. For the second claim that $J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r})$, fix a good $k \in J^i(\boldsymbol{p})$. Then we have $\frac{v_k^i}{r_k} \geq \frac{v_k^i}{p_k} \geq \frac{v_j^i}{p_j} = \frac{v_j^i}{r_j} \geq \frac{v_l^i}{r_l}$ for all goods $l \in [n]_0$. The first inequality holds due to $r_k \leq p_k$, and the second and third inequalities follow from the fact that i demands k at $\boldsymbol{p}$ and j at $\boldsymbol{r}$. Hence, if the buyer demands good k at $\boldsymbol{p}$, then they demand k at $\boldsymbol{r}$.

Now suppose that the buyer demands $j \in B$ at $\boldsymbol{r}$. The proof of the first claim is identical to the case $j \in A$. We prove the last claim that $J^i(\boldsymbol{p}') \subseteq B$. Suppose, for contradiction, that i demands a good $k \in A$ at $\boldsymbol{p}'$, and good $j \in B$ at $\boldsymbol{r}$. This implies $\frac{v_k^i}{p_k'} < \frac{v_k^i}{p_k} = \frac{v_k^i}{r_k} \leq \frac{v_j^i}{r_j} = \frac{v_j^i}{p_j'}$, in contradiction to the fact that k is demanded at $\boldsymbol{p}'$. ☐

We can now prove Lemma 2.

Proof (Proof of Lemma 2). Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be two feasible prices and $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{p}'$ denote their element-wise minimum. As above, π and π' are valid allocations at $\boldsymbol{p}$ and $\boldsymbol{p}'$, and τ is defined as in (3). First we see that τ is indeed an allocation, as $\tau(i,j) \geq 0$ for all buyers i and goods j. It remains to prove that τ satisfies the three criteria of Definition 1 that define validity. We can partition buyers into two sets: the set $\mathcal{B} \subseteq [m]$ of buyers that demand a good in B at $\boldsymbol{r}$, and the set $\mathcal{A} = [m] \setminus \mathcal{B}$ of buyers that do not. Note that, by Lemma 3, the buyers in $\mathcal{B}$ demand only goods in B at $\boldsymbol{p}'$ and, by definition, the buyers in $\mathcal{A}$ demand only goods in A at $\boldsymbol{p}$. It follows that, under π, each buyer $i \in \mathcal{A}$ is only allocated quantities of goods in A (so $\pi(\mathcal{A}, j) > 0$ only if $j \in A$), and, similarly, every buyer $i \in \mathcal{B}$ only receives quantities of goods in B under π' (so $\pi(\mathcal{B}, j) > 0$ only if $j \in B$).

First we show that τ is supply-feasible at $\boldsymbol{r}$. Recall that τ is supply-feasible if $\tau([m], j) \leq s_j$ for all $j \in [n]$. Fix some good $j \in A$. Then by definition of τ, we have $\tau([m], j) = \pi(\mathcal{A}, j) + \pi'(\mathcal{B}, j)$. Recalling that $\pi'(\mathcal{B}, j) = 0$ and making use of the validity of π at $\boldsymbol{p}$, we get $\tau([m], j) \leq \pi([m], j) \leq s_j$. This immediately implies that τ satisfies part (i) of Definition 1 at $\boldsymbol{r}$ for all goods $j \in A$. Analogously, we can show that τ satisfies (i) at $\boldsymbol{r}$ for all goods $j \in B$ by recalling that $\pi(\mathcal{A}, j) = 0$ for any good $j \in B$. As $A \cup B = [n]$, we have shown that τ is supply-feasible.

Next we argue that τ is budget-feasible at $\boldsymbol{r}$. Indeed, note that any buyer in $\mathcal{A}$ only demands goods in A. Moreover, the prices of goods in A are the same at $\boldsymbol{p}$ and $\boldsymbol{r}$, by construction of $\boldsymbol{r}$. It follows that the buyer spends the same under τ at $\boldsymbol{r}$ as they do under π at $\boldsymbol{p}$. As π is budget-feasible at $\boldsymbol{p}$, we see that τ satisfies part (ii) of Definition 1 for all buyers in $\mathcal{A}$. Similarly, as the buyers in $\mathcal{B}$ only demand goods in B, we apply the same argument to see that τ satisfies part (ii) for all buyers in $\mathcal{B}$. As $[m] = \mathcal{A} \cup \mathcal{B}$, τ is budget-feasible.

Finally, we show the demand-validity of τ at $\boldsymbol{r}$. Lemma 3 and the validity of π at $\boldsymbol{p}$ together imply that, for any $j \in [n]$, we have $\tau(i,j) = \pi(i,j) > 0$ only if $j \in J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r})$, which implies that τ satisfies part (iii) of Definition 1 at $\boldsymbol{r}$ for all buyers in $\mathcal{A}$. Analogously, as $\tau(i,j) = \pi'(i,j)$ for all buyers $i \in \mathcal{B}$, we see that τ satisfies constraint (iii) at $\boldsymbol{r}$ for all buyers in $\mathcal{B}$. Hence, τ is demand-valid at $\boldsymbol{r}$, and thus valid. ☐

Corollary 1. *There exists an elementwise-minimal price vector $\boldsymbol{p}^*$.*

Proof. Suppose there exists no such price vector. This means that for all feasible $\boldsymbol{p}$, there exists some feasible $\boldsymbol{q}$ with $q_j < p_j$ for at least one good $j \in [n]$. Fix some feasible prices $\boldsymbol{p}$ with the property that $\boldsymbol{p}$ cannot be reduced any further in any direction without breaking feasibility. Such a point must exist, as the feasible region is closed and restricted to $\mathbb{R}_+^n$. By assumption, there exists a feasible price vector $\boldsymbol{q}$ with $q_j < p_j$ for some $j \in [n]$. Now consider $\boldsymbol{p}' = \boldsymbol{p} \wedge \boldsymbol{q}$. By Lemma 2, $\boldsymbol{p}'$ is feasible. But as $\boldsymbol{p}' \leq \boldsymbol{p}$ with $p_j' < p_j$, this contradicts our assumption that $\boldsymbol{p}$ cannot be reduced further. ☐

4.2 Maximising revenue

In Section 4.1, we established that the set of feasible prices contains a unique elementwise-minimal price vector $\boldsymbol{p}^*$. We now show that revenue is maximised at these prices. Note that we do not assume $\boldsymbol{p}^*$ to be the *only* prices at which revenue is maximised; indeed, there can be many (envy-free) revenue-maximising prices. However, $\boldsymbol{p}^*$ maximises the quantities of goods allocated, and is thus optimal for buyers among all revenue-maximising prices.

Lemma 4. *For any two distinct feasible price vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ with $\boldsymbol{p} \leq \boldsymbol{p}'$ we have $R(\boldsymbol{p}) \geq R(\boldsymbol{p}')$. In other words, the maximum obtainable revenue at $\boldsymbol{p}$ is weakly greater than the revenue obtainable at $\boldsymbol{p}'$.*

Proof. Let π and π' be valid allocations that respectively maximise revenue at $\boldsymbol{p}$ and $\boldsymbol{p}'$. Our goal is to determine a valid allocation τ that achieves a weakly greater revenue at $\boldsymbol{p}$ than π' does at $\boldsymbol{p}'$. As $R(\boldsymbol{p}) \geq \sum_{j\in[n]} p_j \tau([m], j)$ and $R(\boldsymbol{p}') = \sum_{j\in[n]} p'_j \pi'([m], j)$, this immediately implies the result.

If $\boldsymbol{p} = \boldsymbol{p}'$, there is nothing to prove. Hence we assume that $S := \{j \in [n] \mid p_j < p'_j\}$, the set of goods which are priced strictly lower at $\boldsymbol{p}$ than at $\boldsymbol{p}'$, is non-empty. Fix a buyer $i \in [m]$. In order to define the new allocation $\tau(i, \cdot)$ to i at $\boldsymbol{p}$, we distinguish between the two cases that $J^i(\boldsymbol{p})$ is, and is not, a subset of S.

Case 1: Suppose buyer i demands a subset of S at $\boldsymbol{p}$, so $J^i(\boldsymbol{p}) \subseteq S$. In this case, we set $\tau(i, \cdot) = \pi(i, \cdot)$. As π is valid, and τ and π are both allocations at the same prices, the buyer spends its entire budget under τ. Moreover, they are only allocated goods that they demand.

Case 2: Suppose $J^i(\boldsymbol{p}) \not\subseteq S$. We note that $J^i(\boldsymbol{p}') \cap S = \emptyset$ and buyer i still demands all goods in $J^i(\boldsymbol{p}')$, so $J^i(\boldsymbol{p}') \subseteq J^i(\boldsymbol{p})$. In this case, we set $\tau(i, \cdot) = \pi'(i, \cdot)$. As the buyer is only allocated goods not in S, and $p_j = p'_j$ for all goods $j \in [n]_0 \setminus S$, it follows that the buyer spends the same at both prices.

Note that, in both cases, the buyer is only allocated goods that they demand. To summarise, we define τ as

$$\tau(i, \cdot) = \begin{cases} \pi(i, \cdot) & \text{if buyer } i \text{ demands a subset of } S \text{ at } \boldsymbol{p}, \\ \pi'(i, \cdot) & \text{otherwise.} \end{cases}$$

We now prove that τ is valid. We have already argued above that all buyers satisfy the demand and budget conditions of Definition 1. It remains to show that aggregate demand does not exceed supply s_j for any goods $j \in [n]_0$ under τ. Note that a buyer is allocated a good j in S under τ if, and only if, they satisfy Case 1 above. Indeed, in this case we set $\tau(i, j) = \pi(i, j)$. Hence, for any $j \in S$, we have $\tau([m], j) \leq \pi([m], j) \leq s_j$. Similarly, for any $j \notin S$ the buyer will satisfy Case 2, and we get $\tau([m], j) \leq \pi'([m], j) \leq s_j$.

Finally, we see that τ achieves weakly greater revenue at $\boldsymbol{p}$ than π' does at $\boldsymbol{p}'$. To see this, note that each buyer satisfying Case 1 spends its entire budget on non-null goods and thus contributes a weakly greater amount to revenue at $\boldsymbol{p}$ than at $\boldsymbol{p}'$, while a buyer satisfying Case 2 contributes the same amount to revenue. □

Corollary 2. *Revenue is maximised at elementwise-minimal feasible prices $\boldsymbol{p}^*$.*

Proof. Suppose $\boldsymbol{p} \neq \boldsymbol{p}^*$ is a revenue-maximising price vector. As $\boldsymbol{p}^*$ is elementwise-minimal, we have $\boldsymbol{p}^* \leq \boldsymbol{p}$. Then by Lemma 4, we can obtain weakly greater revenue at $\boldsymbol{p}^*$ than at $\boldsymbol{p}$, or $R(\boldsymbol{p}^*) \geq R(\boldsymbol{p})$. But as $\boldsymbol{p}$ maximises R, so does $\boldsymbol{p}^*$. □

5 A price-scaling algorithm

Suppose $\boldsymbol{p}$ is a feasible price vector at which no allocation exhausts supply $\boldsymbol{s}$ in all goods. In this case, we present a method to scale a subset of goods uniformly in price by a factor $0 < c < 1$ while retaining feasibility and increasing aggregate demand. As stated in Corollary 3, this implies that the elementwise-minimal prices $\boldsymbol{p}^*$ clear the market. We do this by modifying a price-scaling routine that forms part of a 'simplex-like' algorithm by Adsul et al. (2012) for Fisher markets to our market setting; this is not immediate, as buyers can choose not to spend their entire budget if prices are too high. In our approach, we scale prices down instead of up, and apply additional pre-processing for bids that may be indifferent between spending on the null good and 'real' goods.

5.1 The procedure for reducing prices

In the following, we describe our subroutine that, given prices $\boldsymbol{p}$ that are not market-clearing and an expenditure e, either modifies e in order to ensure that an allocable good becomes depleted, or reduces prices of some goods. Moreover, we claim that after $O(n)$ repetitions of this subroutine, prices of some goods are reduced. Indeed, we start with at most n allocable goods. Each application of the subroutine that fails to reduce prices strictly reduces the number of allocable goods, as no depleted goods can revert to being allocable. In particular, this implies that the elementwise-minimal prices $\boldsymbol{p}^*$ clear the market, as we could otherwise reduce $\boldsymbol{p}^*$ in price while maintaining feasibility.

Corollary 3. *The elementwise-minimal feasible prices $\boldsymbol{p}^*$ clear the market.*

In order to describe our method of scaling prices, we first introduce the demand and expenditure graphs. Then we present a subroutine that – given prices $\boldsymbol{p}$, an expenditure e and an allocable good j under e at $\boldsymbol{p}$ – either returns scaled-down prices or an expenditure at $\boldsymbol{p}$ under which j is depleted. After invoking this subroutine $O(n)$ times, prices will be scaled down or all goods are depleted. We note that the subroutine defined here is also called in our algorithm (cf. Section 6) to find $\boldsymbol{p}^*$.

5.2 The demand and expenditure graphs

In order to describe demand and spending relationships between bids and goods, we introduce two bipartite graphs, the *demand graph* and the *expenditure graph*. Modulating expenditures in appropriately defined sub-trees of these graphs is the central component in our price reduction procedure.

Definition 4 (Demand graph and expenditure graph). *Let e be a valid expenditure at feasible prices $\boldsymbol{p}$. We denote by $\mathcal{D}$ the bipartite demand graph on vertex sets $[m]$ and $[n]_0$. There is an edge $(i,j) \in D$ if, and only if, buyer i demands j at $\boldsymbol{p}$. Moreover, we define the expenditure graph $\mathcal{E}$ of $\mathcal{D}$ induced by the edges (i,j) with positive expenditure $e(i,j) > 0$. Finally, $\mathcal{D}_j$ and $\mathcal{E}_j$ respectively denote the connected component of $\mathcal{D}$ and $\mathcal{E}$ that contains good j.*

Note that $\mathcal{E}_j$ is a subgraph of $\mathcal{D}_j$ for all goods $j \in [n]_0$, and each $\mathcal{D}_j$ can contain multiple connected components of $\mathcal{E}$. Both graphs are illustrated in Fig. 3.

Breaking cycles. Our price reduction method assumes that the demand graph is acyclic. For this reason, we follow Orlin (2010) in perturbing buyer valuations by an infinitesimal amount in order to break cycles. Induce an ordering $[m] = \{i^1, i^2, \ldots\}$ on the buyers, and perturb each entry $v_j^{i^k}$ of buyer i^k by adding $\varepsilon^{kn} + \varepsilon^j$ for some infinitesimally small $\varepsilon > 0$. In our algorithms, this perturbation can be simulated without a running-time penalty by implementing lexicographic tie-breaking when constructing the demand and expenditure graphs. For details about the perturbation, we refer to (Orlin, 2010).

5.3 Reducing prices along a tree

We now describe the subroutine referred to above. The full procedure is stated in Algorithm 1. Correctness is proved in Theorem 4.

Identifying goods to reduce in price. Suppose j is an allocable good under expenditure e at $\boldsymbol{p}$. We describe a procedure which either identifies a set $J \subseteq [n]$ of goods to reduce in price, or finds a new expenditure e' at $\boldsymbol{p}$ under which good j is depleted. We first construct a directed tree T by rooting the (undirected) tree $\mathcal{D}_j$ at good j and then remove all subtrees rooted at endpoint goods j of arcs $(i,j) \in \mathcal{D}_j \setminus \mathcal{E}$. This is illustrated in Fig. 3. For notational convenience, we also let T_v denote the subtree of T rooted at v for any vertex $v \in T$.

If T does not contain the null good 0, we let J consist of the goods of T and are done. Otherwise, we consider the directed path $(j = g^1, b^1, \ldots, g^k, b^k, g^{k+1} = 0)$ from good j to null good 0 in T and modify the expenditure e to redirect spending from 0 to j as follows. Let $m > 0$ be the minimum of $p_j s_j - e([m], j)$ and

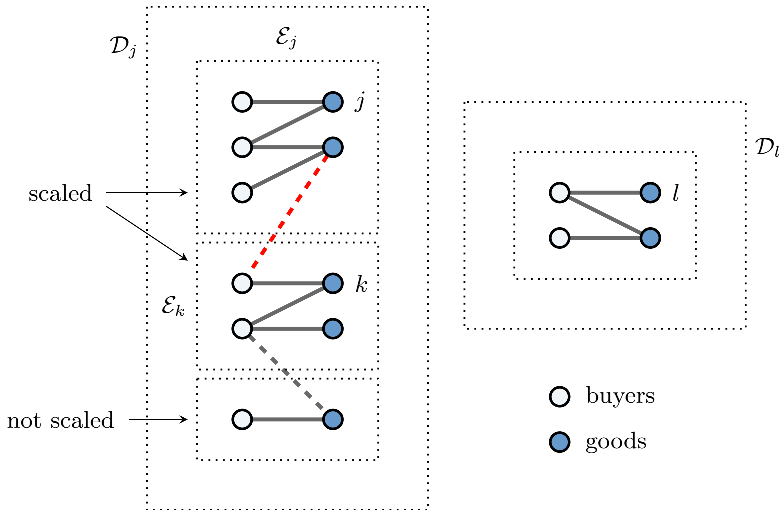

Fig. 3: An illustration of the demand and expenditure graphs. Solid and dashed lines respectively indicate edges in $\mathcal{E}$ and $\mathcal{D} \setminus \mathcal{E}$. The component $\mathcal{D}_i$ contains three components of $\mathcal{E}$. If we root $\mathcal{D}_j$ at j, we note that there is an arc (dashed red) from a good in $\mathcal{E}_j$ to a buyer in $\mathcal{E}_k$, and an arc (dashed grey) from a buyer in $\mathcal{E}_k$ to the component below. Hence the tree constructed in Section 5.3, which contains all goods to be scaled down in price, consists of $\mathcal{E}_j$ and $\mathcal{E}_k$ but not the third component.

Algorithm 1 Tree-based price reduction

Input: Feasible prices $\boldsymbol{p}$, valid expenditure e and allocable good j.

Output: Prices $\boldsymbol{p}' \leq \boldsymbol{p}$ and expenditure e', where either $\boldsymbol{p}' = \boldsymbol{p}$ and j is depleted, or $p'_k < p_k$ is for some goods $k \in [n]$.

1: Construct tree T by rooting $\mathcal{D}_j$ at j and removing subtrees rooted at goods k of arcs $(i,k) \in \mathcal{D}_j \setminus \mathcal{E}$.
2: **if** $0 \notin T$ **then** go to step 9. **end if**
3: Compute the path $j = g^1, b^1, \ldots, g^k, b^k, g^{k+1} = 0$ from j to null good 0 in T and the minimum m of $p_j s_j - e([m], j)$ and $\min_{i \in [k]} e(g^i, \boldsymbol{b}^i)$. Update e by $e(\boldsymbol{b}^i, g^i) \mathrel{+}= m$ and $e(\boldsymbol{b}^i, g^{i+1}) \mathrel{-}= m$.
4: **if** good j is now depleted under e **then**
5:     **return** $\boldsymbol{p}$ and e.
6: **else**
7:     Let i be the smallest such value $i \in [k]$ and remove from T the subtree rooted at g^{i+1}.
8: **end if**
9: Traverse T depth-first to recursively compute $\sum_{k \in T_v} \alpha_k p_k s_k$ for each vertex $v \in T$.
10: Compute $c^{(1)}, c^{(2)}$ and $c^{(3)}$ as specified in Section 5.3, and scaling factor $c^* = \max\left\{c^{(1)}, c^{(2)}, c^{(3)}\right\}$.
11: Traverse T depth-first to recursively compute $\Delta(i,k)$ for each arc in T. Compute e' as described in (4).
12: Let $p'_k = c^* p_k$ for all $k \in T$ and $p'_k = p_k$ otherwise.
13: **return** $\boldsymbol{p}'$ and e'.

$\min_{i \in [k]} e(b^i, g^i)$. We let $e'(b^i, g^i) = e(b^i, g^i) + m$ and $e'(b^i, g^{i+1}) = e(b^j, g^i) - m$ for all $i \in [k]$. Note that the resulting expenditure is valid and redirecting spending in this way does not add edges to the expenditure graph, or cause the expenditure on any goods in $[n]$ to reduce. In particular, no depleted good becomes allocable.

If $m = p_j s_j - e([m], j)$, then good j is depleted under the new expenditure e', and we are done. Otherwise, if $m < p_j s_j - e([m], j)$, we obtain the tree corresponding to the new expenditure e' by identifying the smallest $i \in [k]$ such that $e'(b^i, g^{i+1}) = 0$ and removing the subtree rooted at g^{i+1} from T. As the tree no longer contains the null good 0, we let J denote the goods in T and are done.

Scaling prices. Now we suppose that we have identified the set of goods J of the tree T not containing 0, and good j is allocable. In order to fully specify our scaling method, it remains to determine the factor $0 < c < 1$ by which to uniformly scale the prices of goods in J so that the resulting prices remain feasible. Let $\boldsymbol{p'}$ denote the prices after scaling, so $p'_k = cp_k$ for $k \in J$ and $p'_k = p_k$ otherwise. We proceed by describing an expenditure e' at $\boldsymbol{p'}$ that is derived from e by adding, or removing, a difference term parameterised by c to, or from, each $e(i,k)$. The scaling factor c is then chosen to ensure that e' is valid. Recall that e' needs to satisfy the conditions in Definition 1 to be a valid expenditure at prices $\boldsymbol{p'}$. We recall, for instance, that the supply constraint imposes $e'([m],k) \leq p'_k s_k$ for all goods $k \in [n]$. In the following, we will impose a stronger supply constraint for all goods $k \in [n] \setminus \{j\}$. Suppose that the proportion of good k's supply aggregately demanded under e at $\boldsymbol{p}$ is given by $\alpha_k \in [0,1]$, so that $e([m],k) = \alpha_k p_k s_k$ for good k. Then we require that this proportion is maintained under e' at the new prices $\boldsymbol{p'}$, so $e'([m],k) = \alpha_k p'_k s_k$ for all $k \in [n] \setminus \{j\}$. (Hence $e'([m],k) = c\alpha_k p_k s_k$ for $k \in J$ and $e'([m],k) = \alpha_k p_k s_k$ for $k \notin J$.) Note that this trivially implies the supply constraint, as $\alpha_k \leq 1$ and $c < 1$.

The stronger supply condition uniquely determines an expenditure e' for any given c, which can be expressed by adding, or subtracting, the differential terms $\Delta(i,k)$ to, or from, $e(i,k)$. For every arc in T between some buyer i and good k with endpoint $v \in \{i,k\}$, we define $\Delta(i,k) := (1-c)\sum_{l \in T_v} \alpha_l p_l s_l$. For all other buyers i and k, we let $\Delta(i,k) = 0$. Finally, we define the new expenditure e' as

$$e'(i,k) = \begin{cases} e(i,k) - \Delta(i,k), & \text{if } i \text{ precedes } k \text{ in } T, \\ e(i,k) + \Delta(i,k), & \text{otherwise.} \end{cases} \tag{4}$$

It remains to find the smallest $0 < c < 1$ such that e' is valid. Conceptually, we choose c by reducing from 1 until one of three scenarios occurs. For each scenario $i\{1,2,3\}$, we determine the scaling factor $c^{(i)}$ at which this scenario occurs. Our scaling factor is then chosen by computing $c^* = \max\{c^{(1)}, c^{(2)}, c^{(3)}\}$. The scenarios are as follows.

1. Spending $e([m],j)$ on good j increases until it becomes depleted. This strictly reduces the number of allocable goods in $\mathcal{D}$.
2. Some $e'(i,k)$ decreases to 0. This breaks up a component in $\mathcal{E}$.
3. Some bid not in T starts demanding a good in T (along with goods outside T). This connects two components in $\mathcal{D}$.

Scenario 1: We note that c must be chosen such that $e'([m],j) = e([m],j) + \Delta([m],j) \leq cp_j s_j$. Substituting $\Delta([m],j) = \sum_{k \in J \setminus \{j\}} (1-c)\alpha_k p_k s_k$ and solving for c, we see that this holds for scaling factor

$$c^{(1)} = \frac{e([m],j) + \sum_{k \in J \setminus \{j\}} \alpha_k p_k s_k}{p_j s_j + \sum_{k \in J \setminus \{j\}} \alpha_k p_k s_k}. \tag{5}$$

As $0 \leq e([m],j) \leq p_j s_j$, factor $c^{(1)}$ is well-defined and $0 < c^{(1)} < 1$.

Scenario 2: Note that $\Delta(i,k) > 0$ for any scaling factor c that is strictly less than 1. Hence, $e'(i,k)$ is non-negative for all arcs from a good k to a buyer i, for any $c < 1$. Now fix an arc from buyer i to good k. In this case, we need to ensure that $\Delta(i,k) \leq e(i,k)$ to guarantee $e'(i,k) \geq 0$. Substituting $\Delta(i,k) = (1-c)\sum_{l \in T_k} \alpha_l p_l s_l$, and solving for c, we get

$$c \geq 1 - \frac{e(i,j)}{\sum_{k \in T_j} \alpha_k p_k s_k}. \tag{6}$$

Hence $c^{(2)}$, as chosen below, guarantees $e'(i,j) \geq 0$ for every buyer i and good j, with equality for at least one pair (i,j).

$$c^{(2)} = \max_{(i,k) \in T} \left(1 - \frac{e(i,k)}{\sum_{l \in T_k} \alpha_l p_l s_l}\right). \tag{7}$$

Note that $0 < e(i,k) \le \alpha_k p_k s_k \le \sum_{l \in T_k} \alpha_l p_l s_l$ for all $(i,k) \in T$, so $c^{(2)}$ is well-defined and satisfies $0 < c^{(2)} < 1$.

Scenario 3: Fix a buyer i not in T. Then the buyer continues to demand goods outside T only if $\frac{1}{c} \max_{k \in T} \frac{v_k^i}{p_k} \le \max_{l \notin T} \frac{v_l^i}{p_l}$. Hence, by solving for c and taking the maximum over all bids not in T, we get

$$c^{(3)} = \max_{i \notin T} \left[\max_{k \in T} \frac{v_k^i}{p_k} \min_{l \notin T} \frac{p_l}{v_l^i} \right]. \tag{8}$$

Note that $1 \le \max_{k \in T} \frac{v_k^i}{p_k} < \max_{l \in [n]_0} \frac{v_l^i}{p_l}$ for all $i \notin T$, as these bids do not demand any goods in T at $\boldsymbol{p}$. Hence, $c^{(3)}$ is well-defined and $0 < c^{(3)} < 1$.

Theorem 4. *Suppose $\boldsymbol{p}$ is not market-clearing, and e is a valid expenditure at $\boldsymbol{p}$ with an allocable good j. The price-reduction procedure above (Algorithm 1), when given $\boldsymbol{p}$, e and j, either returns a valid expenditure e' at $\boldsymbol{p}$ that sates good j, or it returns reduced prices $\boldsymbol{p}'$ and a valid expenditure e' at $\boldsymbol{p}'$, in polynomial time.*

Proof. Suppose first that the subroutine terminates in line 5. Then the resulting expenditure e' that it returns is trivially valid, and we are done.

Now suppose that the subroutine does not terminate in line 5. Then we note that $\boldsymbol{p}'$ returned by the procedure is strictly smaller than $\boldsymbol{p}$ for all goods $k \in J$, as the scaling factor is strictly less than 1. Hence it suffices to show that the expenditure e' returned by Algorithm 1 after scaling prices of goods J by factor c^* is valid in the sense of Definition 1. By construction of c^*, $e'(i,k)$ is non-negative for all buyers $[m]$ and goods k. We also note that all arcs in T coincide with edges in the demand graph by construction, and if $e(i,k)$ changes, then there is an edge between i and k in the demand graph. Meanwhile, the choice of $c^{(3)}$ guarantees that all buyers $i \notin T$ demand, at $\boldsymbol{p}'$, a superset of the goods they demand at $\boldsymbol{p}$. It follows e is demand-valid.

Next, we verify that e is budget-feasible. Suppose first that i is a buyer not in T. Then, by construction of T, none of the prices of goods that i demands are scaled, and the spending of i is unchanged. Next suppose that buyer i is a leaf in T, and good k is their parent. Then $\Delta(i,k) = \sum_{l \in T_i} \alpha_l p_l s_l = 0$, as the tree T_i contains no goods. Hence, $e'(i,k) = e(i,k) = \beta^i$ and the condition for budget feasibility holds. Finally, suppose buyer i is not a leaf in T. Let g^1 be their parent and $g^2, \ldots, g^k$ be the children. We need to verify that $e'(i,[n]_0) = \beta^i$. As $e(i,[n]_0) = \beta^i$, it suffices to see that $-\Delta(i,g^1) = \sum_{j=2}^k \Delta(i,g^j)$. But this follows from the definition of Δ.

Next, we show that e is supply-feasible. For the originally allocable good j, this follows immediately from the choice of our scaling factor $c^* \ge c^{(1)}$ in (5). For all other goods, we verify the stronger supply condition introduced above. Suppose first that k is a good not in T. Then by construction of T, the amount that each bid spends on k remains unchanged. Next suppose good k is a leaf in T, and bid i is its parent. Then the stronger supply constraint requires that $e'(i,k) = ce(i,k) = c\alpha_k p_k s_k$, which agrees with our definition of $\Delta(i,k) = (1-c)\alpha_k p_k s_k$. Finally, suppose good k is not a leaf in T, and let i be its parent and $[m]'$ be its children. Then the stronger supply constraint is satisfied, as

$$e'([m],k) = e([m],k) - \Delta(i,k) + \Delta([m]',k) = \alpha_k p_k s_k - (1-c)\alpha_k p_k s_k = c\alpha_k p_k s_k.$$

To see that the subroutine runs in polynomial time, note that the running time is dominated by traversing the tree with $n+m$ nodes depth-first, which is polynomial in n and m. □

6 Clearing the market

The simplex-like algorithm of Adsul et al. (2012) for linear Fisher markets proceeds by repeatedly applying a price-scaling subroutine until it returns the unique market-clearing prices. Similarly, we can repeatedly apply our subroutine from Section 5 in order to find market-clearing prices $\boldsymbol{p}^*$ in our market in finite time. We show in Proposition 5 that these prices uniquely clear the market. Note that each call to Algorithm 1 makes progress in the sense that prices decrease or a good is depleted. Moreover, the running time analysis

Algorithm 2 Finding elementwise-minimal feasible prices $\boldsymbol{p}^*$

Input: List of buyers and their valuations in general position.
Output: Element-wise smallest feasible prices $\boldsymbol{p}'$.

Let $\boldsymbol{p}$ be the element-wise maximum over the buyers' valuations in $[m]$ and e be the expenditure at which all buyers spend their entire budget on the null good 0.
while there are allocable goods **do**
    Identify an allocable good j under e at $\boldsymbol{p}$.
    while good j is allocable **do**
        Apply tree-based subroutine stated in Algorithm 1 with good j to update $(\boldsymbol{p}, e)$.
    end while
end while
return prices $\boldsymbol{p}$ and expenditure e.

we provide in Theorem 6 is analogous to that given in (Adsul et al., 2012), and shows that the algorithm terminates in at most exponential time.

Let $\boldsymbol{p} = \bigvee_{i\in[m]} \boldsymbol{v}^i$ be the element-wise maximum over all valuation vectors. Then it follows that all buyers demand the null good at $\boldsymbol{p}$, while possibly being indifferent between null and other goods. We define the expenditure e at $\boldsymbol{p}$ under which all buyers spend their entire budget on the null good, so for each buyer $i \in [m]$ we let $e(i, 0) = \beta^i$ and $e(i, j) = 0$ for all $j \in [n]$. This expenditure is trivially valid.

Of course, the market does not clear at this initial allocation. We pick any allocable good j and repeatedly call the subroutine of Algorithm 1 with good j, prices $\boldsymbol{p}$ and expenditure e to reduce prices until the market for good j clears (good j becomes depleted). Subsequently, we select another allocable good, reduce prices by repeatedly calling Algorithm 1, and repeat this step until all goods are depleted. In order to argue that the prices found correspond to $\boldsymbol{p}^*$, we prove that market-clearing prices are unique in our market setting.

Proposition 5. *Market-clearing prices are unique.*

Proof. Suppose we have a allocation π exhausting supply $\boldsymbol{s}$ that is valid at some prices. We want to find these corresponding prices $\boldsymbol{p}$ that support π. First, we formulate the problem of finding $\boldsymbol{p} \geq 0$ as a system of linear inequalities by writing out the budget constraints as equalities and the demand constraints as inequalities. The latter is achieved by adding, for every buyer i who demands a positive amount of good j, the inequalities $\frac{v^i_j}{p_j} \geq \frac{v^i_k}{p_k}, \forall k \in [n]_0$. Our goal is to prove that there is exactly one solution to this system of inequalities.

Corollary 3 tells us that the elementwise-minimal feasible prices $\boldsymbol{p}^*$ are market-clearing, so the system has at least one solution. For contradiction, suppose there is a second feasible price vector $\boldsymbol{p} \geq \boldsymbol{p}^*$ that also clears the market. Then by convexity of the polytope, it follows that $\boldsymbol{p}' = \boldsymbol{p}^* + \varepsilon(\boldsymbol{p} - \boldsymbol{p}^*)$ is also market-clearing for any $0 \leq \varepsilon \leq 1$. From now on, we assume that $\boldsymbol{p}'$ is obtained by letting ε be infinitesimally small. Let $S := \{j \in [n] \mid p^*_j < p_j\}$ denote the prices that change when we move from $\boldsymbol{p}^*$ to $\boldsymbol{p}'$. As $\boldsymbol{p} \neq \boldsymbol{p}^*$, S is non-empty. We first make a technical observation about the demand of each bid at $\boldsymbol{p}^*$ and $\boldsymbol{p}'$.

Observation 5. *For any buyer $i \in [m]$, we note: (i) If $J^i(\boldsymbol{p}^*) \not\subseteq S$, then $J^i(\boldsymbol{p}') = J^i(\boldsymbol{p}^*) \setminus S$. (ii) If $J^i(\boldsymbol{p}^*) \subseteq S$, then $J^i(\boldsymbol{p}') \subseteq J^i(\boldsymbol{p}^*)$. This holds because ε is chosen to be infinitesimally small.*

Now suppose e' is a valid expenditure at $\boldsymbol{p}'$ that clears the market. Let $\mathcal{S} := \{i \in [m] \mid J^i(\boldsymbol{p}^*) \subseteq S\}$ denote the set of buyers satisfying case (ii) of Observation 5. We take a look at the revenue contributions from each buyer towards goods in S. At $\boldsymbol{p}'$, Observation 5 tells us that only buyers $i \in \mathcal{S}$ spend on goods in S, and they each spend their entire budget on S. Hence, the budget and supply feasibility of e' imply $\sum_{i\in\mathcal{S}} \beta^i = e'(\mathcal{S}, S) = e'([m], S) = \sum_{j\in S} p'_j s_j$. At $\boldsymbol{p}^*$, the buyers of $\mathcal{S}$ also spend their entire budget on S, but it may be the case that other buyers also spend part of their budget on S. Hence, $\sum_{i\in\mathcal{S}} \beta^i = e(\mathcal{S}, S) \leq e([m], S) = \sum_{j\in S} p^*_j s_j$. It follows that $\sum_{j\in S} p'_j s_j \leq \sum_{j\in S} p^*_j s_j$, in contradiction to the fact that $p'_j > p^*_j$ for all $j \in S$. ◻

Our algorithm can be used in practice to find competitive equilibrium prices in exponential time. To the best of our knowledge, our paper provides the first combinatorial algorithm for solving for competitive equilibrium prices in quasi-Fisher markets.[10] For arctic auctions, previous (exponential time) algorithms (Fichtl, 2022) could only find some revenue optimal prices, but not necessarily those that also support a competitive equilibrium.

Theorem 6. *Algorithm 2 finds the market-clearing prices $\boldsymbol{p}^*$ in at most exponential time.*

Proof. We argue that the number of calls to Algorithm 1 is bounded. Note that the outer loop in Algorithm 2 iterates $O(n)$ times. Indeed, invoking Algorithm 1 does not cause any depleted goods to become allocable. Every iteration of the outer loop reduces the number of allocable goods by one, and we start with at most n allocable goods.

We now show that the inner loop of Algorithm 2 applies Algorithm 1 an exponential number of times (in m and n) before good j is depleted. Suppose, for contradiction, that the subroutine is called more often. In this case, we note that on each iteration, the scaling factor is chosen such that scenario 2, scenario 3 or both scenarios occur. Let $\mathcal{D}_j^{(k)}$ denote the connected component of $\mathcal{D}$ containing good j when scenario 2 occurs for the k-th time. In particular, we note that any two consecutive occurrences of scenario 2 can be separated by at most m iterations of scenario 3's. To see this, realise that every time only scenario 3 occurs, T grows through the addition of a buyer, and we have m buyers in total.

We now argue that all $\mathcal{D}_j^{(l)}$ are distinct. As there are exponentially many configurations of $\mathcal{D}_j$, this implies the result. Note that whenever the k-th scenario 2 occurs, we have $e'(i,k) = 0$ for some arc (i,k), and good $k \in T$ is not in T in the next iteration. Hence, the price of good k is not scaled down in the next iteration. Meanwhile, the price of good j is scaled down on every iteration of Algorithm 1. Now suppose, for any k, that $\mathcal{D}_i^{(k)}$ reoccurs at a later stage. This implies that p_j and p_k have been scaled down the same; indeed, this follows by considering the path from j to k in $\mathcal{D}_j^{(k)}$ and applying the conditions of demand. But we have just argued that p_j and p_k have not been scaled down the same, a contradiction.

Finally, we see that the algorithm finds $\boldsymbol{p}^*$. To see this, note that it terminates only once all goods are depleted. As the elementwise-minimal prices are the unique market-clearing prices (cf. Corollary 3 and Proposition 5), the result follows. ◻

7 Conclusion

We analyse a market for multiple divisible goods, in which a unique set of prices induce a socially optimal and revenue-optimal allocation. This coincidence of revenue-optimality and efficiency makes our market compelling in theory as well as highly attractive for sellers, buyers, and market platforms in practice. We provide algorithmic results to derive the efficient and optimal price vector. Our approach is based on a novel geometric understanding of the structure underlying feasible market prices, which may be of independent interest. Future work includes addressing the open question of whether the revenue-welfare equivalence holds for other classes of preferences, and whether there exists a polynomial bound on the number of steps of our algorithm or variations thereof.

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[10] We say 'combinatorial' algorithm to contrast with the inner-point methods by Chen et al. (2007).

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